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Introducing fuzzy reactive graphs: a simple application on biology

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Abstract

In this paper, we propose a generalization for fuzzy graphs in order to model reactive systems with fuzziness. As we will show, the resulting fuzzy structure, called fuzzy reactive graphs (FRG), is able to model dynamical aspects of some entities which generally appear in: biology, computer science and some other fields. The dynamical aspect is captured by a transition function which updates the values of the graph after an edge has been crossed. The update process takes into account aggregation functions. The paper proposes a notion for bisimulation for such graphs and briefly shows how modal logic can be used to verify properties of systems modeled with FSGs. The paper closes with a toy example in the field of Biology.

Keywords Fuzzy switch graphs · Fuzzy reactive graphs · Bisimulation · Fuzzy graphs · Fuzzy systems · Reactive systems · Biological systems · Synthetic biology

1 Introduction

In the real world, there are state-based systems with fuzzy behavior in which the transition of one state to another entails a system reconfiguration. The process of reconfiguration is called here *reactivity*. The concept of reactivity on state-based transition systems has been introduced by several authors, such as: van Benthem (van Benthem 2005), Areces (Areces et al. 2014, 2015) and Gabbay (Gabbay and Marcelino 2012, 2009). Some of such reactive models propose a non-fixed accessibility relation (set of edges) between states of the systems which vary according to a taken path.

In this paper, we propose models which are able to express the uncertainties (fuzzyness) in systems (biological, computational, etc.); it is called *fuzzy reactive graphs (FRG)*. It extends the notion of fuzzy graphs in the sense that the crossing of an arrow induces an update of the system; namely, the edges are updated according to an *aggregation function* whenever an edge between nodes is crossed. To achieve that, the model takes into account usual edges (from nodes to nodes), called first-order edges, as well as non-first-order edges (called higher-order edges), which are those that connect first-order edges to any kind of edges.

The paper shows the effect of some aggregations on FRGs, introduces the product of FRGs and proposes a notion of bisimulation for them. It ends by providing an application for the biological setting.

The paper is divided into the following sections: Sect. 2 provides some background for this text. Section 3 introduces the notion of *fuzzy reactive graphs*, shows the effect of some aggregations on FRGs and provides the product of FRGs. Section 4 shows a connection between FRGs and fuzzy graphs. Section 5 provides a logic for FRGs as well introduces the notion of bisimulation for them. Section 6 shows how FSGs can be used as an alternative to model some biological systems. Finally, Sect. 7 provides some final comments.

2 Preliminaries

In this section, we introduce few notions related to the fuzzy framework in order to make this paper self-contained. We assume some basic knowledge in Fuzzy sets theory.

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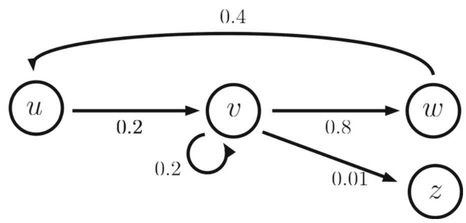


Fig. 1 Fuzzy graph G

Fig. 2 Matrix representation of G. The blank values means zero

	u	v	w	z
u		0.2		
v		0.2	0.8	0.01
w	0.4			
z				

Definition 1 (Beliakov et al. 2007) A fuzzy set A defined on a set of objects X is represented by a **membership function** $\mu_A : X \to [0, 1]$, in such a way that for any object $x \in X$, the value $\mu_A(x)$ measures the degree of membership of x in the fuzzy set A. Here, we use the notation $gr(x) = g$ to mean that $g = \mu_A(x)$.

Definition 2 (Fuzzy Graphs) A **fuzzy graph** (Lee 2006) is a structure: $G = \langle V, R\rangle$, such that V is a set called **set of vertices** and R is **a fuzzy binary relation** on V; namely, a fuzzy set $R : V \times V \to [0, 1]$.

Fuzzy graphs are represented here by using matrices in which the rows and columns are labeled by the elements of V or in a pictorial way.

Example 1 Figure 1 shows a fuzzy graph which will be used across this paper to illustrate some definitions. The corresponding matrix is presented in Fig. 2. For further information on fuzzy graphs, see (Lee 2006).

In this work, we will consider more or less accepted notions of what are fuzzy conjunctions, disjunctions, implications and negations. The first two are generalized by T-norms and T-conorms, respectively (Klement et al. 2013), whereas fuzzy implications and negations has been also widely studied (Baczyński and Jayaram 2008).

Definition 3 (T-norms and T-conorms: Pinheiro et al. 2018) A bivariate function, $U : [0, 1] \times [0, 1] \to [0, 1]$, is called **uninorm** if it is isotonic, commutative, associative with a neutral element $e \in [0, 1]$. If $e = 1$, then U is called **T-norm** and if $e = 0$, then U is called **T-conorm or S-norm**.

Example 2 The functions, $T_M, S_M : [0, 1]^2 \to [0, 1]$, s.t. $T_M(x, y) = \min(x, y)$ and $S_M(x, y) = \max(x, y)$ are a T-norm and T-conorm, respectively. For details, see (Klement et al. 2013).

Definition 4 (Negations: Pinheiro et al. 2018) A unary operation, $N : [0, 1] \to [0, 1]$, is called **fuzzy negation**, if it is antitonic, $N(0) = 1$ and $N(1) = 0$. N is **strong**, whenever $N(N(x)) = x$. For any fuzzy negation, N, and functions: $f, f_N : [0, 1]^n \to [0, 1]$, f_N is called the N**-dual** of f whenever $f_N(x_1, \ldots, x_n) = N(f(N(x_1), \ldots, N(x_n)))$.

Example 3 The function $N_G(0) = 1$ and $N_G(x) = 0$, whenever $x > 0$ is called Gödel Negation.

Definition 5 (Implications: Baczyński and Jayaram 2008; Pinheiro et al. 2018; Reiser et al. Oct 2013; Andrade et al. 2014; Pinheiro et al. 2017) A bivariate function, $I : [0, 1] \times [0, 1] \to [0, 1]$, is called **fuzzy implication** whenever it satisfies the following properties:

1. Corner Condition (CC). $I(1, 0) = 0$ and $I(0, 0) = I(0, 1) = I(1, 1) = 1$,
2. First place antitonicity (FPA). if $x \leq z$, then $I(x, y) \geq I(z, y)$ and
3. Second place isotonicity (SPI). if $y \leq z$ then $I(x, y) \leq I(x, z)$.

A fuzzy implication, I, satisfies the **order property (OP)**, whenever:

$$x \leq y \text{ implies } I(x, y) = 1. \tag{1}$$

Example 4 Gödel Implication: the function, $I_G : [0, 1]^2 \to [0, 1]$,

$$I_G(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases}$$

With respect to bi-implications, there is no universal agreement about a fuzzy counterpart. The most well-known class of fuzzy bi-implications was investigated by Fodor and Roubens (Fodor and Roubens 1994) who suggested the following definition.

Definition 6 (Bi-implications: Callejas et al. 2012; Claudio Callejas 2013) A bivariate function, $B : [0, 1] \times [0, 1] \to [0, 1]$, is called **fuzzy bi-implication** whenever it satisfies the following properties:

1. $B(x, y) = B(y, x)$—B-commutativity.
2. $B(x, x) = 1$—B-identity.
3. $B(0, 1) = 0$—Corner Condition.
4. If $w \leq x \leq y \leq z$, then $B(w, z) \leq B(x, y)$.

Example 5 The function, $B_Z : [0, 1]^2 \to [0, 1]$,

$$B_Z = \begin{cases} 1, & \text{if } x = y \\ \min(x, y), & \text{otherwise} \end{cases}$$

is a fuzzy bi-implication (Claudio Callejas 2013).

Proposition 1 *Given a T-norm, T, and an implication, I, satisfying (OP), then $B(x, y) = T(I(x, y), I(y, x))$ is a fuzzy bi-implication.*

Proof 1. $B(x, y) = T(I(x, y), I(y, x)) = B(y, x)$.
2. $B(0, 1) = T(I(0, 1), I(1, 0)) = T(1, 0) = 0$.
3. $B(x, x) = T(I(x, x), I(x, x)) \overset{(OP)}{=} T(1, 1) = 1$.
4. Suppose $w \le x \le y \le z$, then $B(w, z) = T(I(w, z), I(z, w)) \overset{(OP)}{=} T(1, I(z, w)) = I(z, w) \overset{(SPI)}{\le} I(z, x) \overset{(FPA)}{\le} I(y, x) = T(1, I(y, x)) \overset{(OP)}{=} T(I(x, y), I(y, x)) = B(x, y)$.

□

Corollary 1 *B_G is a bi-implication.*

Proof Since I_G satisfies (OP), then $B_G(x, y) = T_G(I_G(x, y), I_G(y, x))$. □

Definition 7 A structure $\mathcal{F} = \langle [0, 1], T, S, N, I, B, 0, 1\rangle$, s.t. T is a T-norm, S is a T-conorm, N a fuzzy negation, I is a fuzzy implication, and B a fuzzy bi-implication is called **fuzzy semantics** (Cruz et al. 2018).

Example 6 (Gödel Semantics)

$$\mathcal{G} = \langle [0, 1], T_M, S_M, N_G, I_G, B_G, 0, 1\rangle$$

Another important fuzzy entity used in this paper is called aggregation functions or simply *aggregation*. They generalize the well-known means like *arithmetic, weighted and geometric*. T-norms and S-norms are also examples aggregations.

Definition 8 (Aggregation functions: Beliakov et al. 2007; Farias et al. 2016; Costa et al. 2018) An n-ary function $A : [0, 1]^n \to [0, 1]$ is called an **aggregation function** or just **aggregation** if

1. A is isotonic—i.e., for all $i \in \{1, \dots, n\}$, if $x_i \le y_i$, then $A(x_1, \dots, x_n) \le A(y_1, \dots, y_n)$.
2. $A(0, \dots, 0) = 0$ and $A(1, \dots, 1) = 1$.

Additional examples of aggregations are the projection functions: $\pi_j : A_1 \times \cdots \times A_j \times \cdots \times A_n \to A_j$, s.t. $\pi_j(x_1, \dots, x_j, \dots, x_n) = x_j$, which will be used in this paper.

Definition 9 An aggregation A is **conjunctive** if $A(\bar{x}) \le \min(\bar{x})$, **disjunctive** if $\max(\bar{x}) \le A(\bar{x})$ and **average** if $\min(\bar{x}) \le A(\bar{x}) \le \max(\bar{x})$, for every $\bar{x} \in [0, 1]^n$.

An aggregation $A : [0, 1]^n \to [0, 1]$ is **shift-invariant** if, for all $\lambda \in [-1, 1]$ and for all $\bar{x} \in [0, 1]^n$,

$$A(x_1 + \lambda, ..., x_n + \lambda) = A(x_1, ..., x_n) + \lambda$$

whenever $(x_1 + \lambda, ..., x_n + \lambda) \in [0, 1]^n$ and $A(x_1, ..., x_n) + \lambda \in [0, 1]$.

Example 7 T-norms are conjunctive aggregations, T-conorms are disjunctive, and means (arithmetic, geometric and weighted) are average aggregations.

Definition 10 (Beliakov et al. 2007) An element $a \in]0, 1[$ is a zero divisor of an aggregation function A if for all $i \in \{1, \dots, n\}$, there exists some $\mathbf{x} \in]0, 1]^n$ such that its ith component is $x_i = a$, and $A(x) = 0$, i.e., the equality:

$$A(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n) = 0 \tag{2}$$

holds for some $\mathbf{x} > 0$, with a at any position.

In other words, it is possible to obtain the value 0 even for inputs which does not contain zero. Moreover, if a is a zero divisor, then all the values $b < 0$ are also zero divisors. In other words, it works like a threshold.

Example 8 The means (arithmetic, weighted and geometric), T-norms and T-conorms are aggregations. The function: $f(x_1, x_2) = \max(0, x_1 + x_2 - 1)$ is an aggregation with zero divisor $a = 0.999$, provided that the other component is sufficiently small (e.g., 0.0005).

* In this work, we will avoid aggregations with zero divisors, since it will induce de disconnection of edges.

3 Fuzzy switch graphs and reactivity

In this section, we introduce a structure which generalizes the notion of fuzzy graphs (Lee 2004). As we will show, they are very useful to represent mechanisms like some *biological phenomena*.

Definition 11 Let W be a non-empty finite set, whose elements are called **states or worlds**, and following recursive defined family of crisp sets:

$$\begin{cases} S^0 \subseteq W \times W \\ S^{n+1} \subseteq S^0 \times S^n \end{cases} \tag{3}$$

and $S = \bigcup_{i \in \mathbb{N}} S^i$. A **fuzzy switch graph (FSG)** is a pair $M = \langle W, \mu : S \to [0, 1]\rangle$, where $\mu : S \to [0, 1]$ is a fuzzy subset of S. The weighted directed edges, $a^0 \in W \times W$, are called **first-order arrows**, whereas the remaining are called **high-order arrows**. We will denote a FSG simply by $\langle W, \mu\rangle$.

Table 1 Truth values of propositions on each state

	u	v	w	z
u		0.2		
v		0.2	0.8	0.01
w	0.4			
z				

S^0 contains the arrows which relate the states in W, whereas S^{i+1} contains the arrows which relates S^0 to the arrows at S^i. In the following, we make an abuse of notation. We use the notation, a_i^0, for both: $a_i^0 = (x, y) \in S^0$ and $a_i^0 = (x, y, gr(x, y)) \in \mu$. We assume the first notation when the context is clear.

Example 9 Suppose we have a set of states $W = \{u, v, w, z\}$ together with the relation, S^0, given in Table 1.

Make $a_1^0 = (u, v, 0.2), a_2^0 = (v, v, 0.2), a_3^0 = (v, w, 0.8), a_4^0 = (v, z, 0.01), a_5^0 = (w, u, 0.4) \in S^0$. Now, relate the arrows in S^0 and build the *second-order arrows* in S^1 by making: $a_1^1 = (a_2^0, a_4^0, 0.2), a_2^1 = (a_3^0, a_5^0, 0.7) \in S^1$ (c.f. Table 2a). Finally, build the *third-order arrow*, $a_1^2 = (a_1^0, a_2^1, 0.1)$ to relate those in S^0 to those in S^1 (cf. Table 2b).

The resulting fuzzy switch graph can be pictured as in Fig. 3 which represents a system configuration.

Definition 12 Given two FSGs: $M = \langle W, \mu\rangle$ and $N = \langle W, \mu'\rangle$, M is a **subgraph (supergraph) of** N if $\mu \subseteq \mu'$ ($\mu \supseteq \mu'$)—i.e., for all $a \in S$, $\mu(a) \leq \mu'(a)$ ($\mu(a) \geq \mu'(a)$).

Definition 13 Given two FSGs $M = \langle W, \mu\rangle$ and $M' = \langle W, \mu'\rangle$, M' is a **translation of** M by $\lambda \in [-1, 1]$ if for all $a \in S$, s.t. $\mu(a) > 0$, $\mu'(a) = \mu(a) + \lambda$. In this case, M' is written as $\tau(M \mid \lambda) = \langle W, \tau(\mu \mid \lambda)\rangle$ (Fig. 4).

3.1 Reconfiguration and reactivity

The *reactivity* of a system is modeled here in the following way:

Whenever a first-order arrow is crossed, the fuzzy grade of its target arrow is updated.

Figure 5 provides an example of a sequence of configurations using the *arithmetic mean* as aggregation. After crossing $a_1^0 = (u, v, 0.2)$, the arrow $a_2^1 = (a_3^0, a_5^0, 0.7)$ is *updated* to $a_2^1 = (a_3^0, a_5^0, 0.33)$ by using the *arithmetic mean* between the grades of a_1^0, a_2^1 and a_1^2. A second step, by crossing $a_3^0 = (v, w, 0.8)$ *updates*, in the same manner, the arrow $a_5^0 = (w, u, 0.4)$ to $a_5^0 = (w, u, 0.4)$.

The system has a (possibly null) *reactivity* after a change of state. Each step produces a new configuration. This new configuration is defined below:

Definition 14 Given a FSG $M = \langle W, \mu : S \to [0, 1]\rangle$ and an aggregation function without zero divisors, $A : [0, 1]^3 \to [0, 1]$, a **FSG based on** A **after crossing a first-order arrow**, a_i^0, is the FSG $M_A^{a_i^0} = \langle W, \mu_{a_i^0}^A : S \to [0, 1]\rangle$ s.t.

$$\mu_{a_i^0}^A(a) = \begin{cases} \mu(a), & \text{if } (a_i^0, a) \notin S \\ A\Big(\mu(a_i^0), \mu(a_i^0, a), \mu(a)\Big), & \text{otherwise.} \end{cases} \quad (4)$$

The FSG $M_A^{a_i^0} = \langle W, \mu_{a_i^0}^A : S \to [0, 1]\rangle$ is called **reconfiguration of** M, **based on** A, **after crossing** a_i^0.

It is easy to verify that the FSGs in Fig. 5 satisfy the above definition.

Since every aggregation function satisfies $A(1, \ldots, 1) = 1$, then the reconfiguration is innocuous whenever the value of the involved arrows is one.

Proposition 2 *If A is a conjunctive (disjunctive) aggregation and $M = \langle W, \mu\rangle$ is a FSG, then $M_A^{a_i^0}$ is a subgraph (supergraph) of M.*

Proof Case $(a_i^0, a) \notin S$, then trivially $\mu_{a_i^0}^A(a) \leq \mu(a)$. Otherwise, $\mu_{a_i^0}^A(a) \overset{\textbf{def}}{=} A(\mu(a_i^0), \mu(a_i^0, a), \mu(a)) \leq \min(\mu(a_i^0), \mu(a_i^0, a), \mu(a)) \leq \mu(a)$. The dual statement follows straightforwardly. □

Proposition 3 *Given two FSGs: $M = \langle W, \mu\rangle$ and $N = \langle W, \theta\rangle$ s.t. $N = \tau(M \mid \lambda)$. If $A : [0, 1]^3 \to [0, 1]$ is a shift-invariant aggregation and $A(\vec{x}) + \lambda \in [0, 1]$, then $N_{a_i^0}^A = \tau(M_{a_i^0}^A \mid \lambda)$.*

Proof Suppose that $A(\mu(a_i^0), \mu(a_i^0, a), \mu(a)) + \lambda \in [0, 1]$. Case $(a_i^0, a) \in S$, then $\theta_{a_i^0}^A(a) \overset{\textbf{def}}{=} A(\theta(a_i^0), \theta(a_i^0, a), \theta(a)) \overset{\textbf{def}}{=} A(\mu(a_i^0) + \lambda, \mu(a_i^0, a) + \lambda, \mu(a) + \lambda) \overset{\textbf{hip}}{=} A(\mu(a_i^0), \mu(a_i^0, a), \mu(a)) + \lambda = \mu_{a_i^0}^A(a) + \lambda \overset{\textbf{def}}{=} \tau(\mu_{a_i^0}^A(a) \mid \lambda)(a)$. Case $(a^0, a) \notin S$, $\theta_{a_i^0}^A(a) \overset{\textbf{def}}{=} \theta(a) = \mu(a) + \lambda = \mu_{a_i^0}^A(a) + \lambda = \tau(\mu_{a_i^0}^A \mid \lambda)(a)$. Therefore, $N_{a_i^0}^A = \tau(M_{a_i^0}^A \mid \lambda)$. □

Now, we introduce our notion of reactivity.

Definition 15 Let $M = \langle W, \mu : S \to [0, 1]\rangle$ be a FSG, A set of aggregation functions and a function $Ag : S^0 \to A$. The pair $\langle M, Ag\rangle$ is called **fuzzy reactive graph (FRG)**.

Given $a_i^0 \in \Gamma$, the **reconfiguration of** $\langle M, Ag\rangle$ **after crossing** a_i^0 is the FRG $\langle M^{a_i^0}, Ag\rangle$, where $M^{a_i^0} = \langle W, \mu_{a_i^0}^{Ag} :$

Table 2 Relations S^1 and S^2

	a_1^0	a_2^0	a_3^0	a_4^0	a_5^0
a_1^0					
a_2^0				0.2	
a_3^0					0.7
a_4^0					
a_5^0					

(a) Relation S^1 — second-order arrows.

	a_1^1	a_2^1
a_1^0		0.1
a_2^0		
a_3^0		
a_4^0		
a_5^0		

(b) Relation S^2 — third-order arrows.

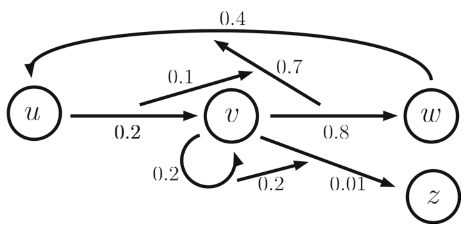


Fig. 3 Graphical representation of the relations in Tables 1 and 2

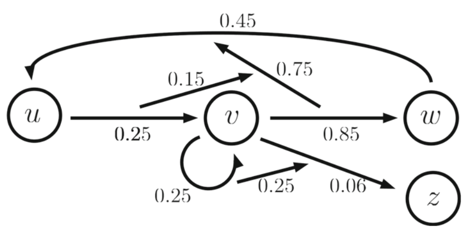


Fig. 4 Translation of graph of Fig. 3 by $\lambda = 0.05$

$S \to [0, 1]\rangle$ is the FSG s.t.

$$\mu^{Ag}_{a_i^0}(a) = \begin{cases} \mu(a), & \text{if}(a_i^0, a) \notin S \\ Ag(a_i^0)\Big(\mu(a_i^0), \mu(a_i^0, a), \mu(a)\Big), & \text{otherwise.} \end{cases} \quad (5)$$

Example 10 Let $M = \langle W, \mu : S \to [0, 1]\rangle$ be the FSG in Fig. 3. Then, $S^0 = \{a_1^0, a_2^0, a_3^0, a_4^0, a_5^0\}$. Making $A = \{arith, \max\}$, $Ag(a_1^0) = Ag(a_2^0) = arith$ and $Ag(a_3^0) = Ag(a_4^0) = Ag(a_5^0) = \max$, the structure $\langle M, Ag\rangle$ is a FRG. Figure 6 contains $\langle M^{a_1^0}, Ag\rangle$ and $\langle M^{a_3^0}, Ag\rangle$, respectively.

In this example, the same FSG is updated by different aggregations (depending on the crossed arrow).

In what follows, we present the Cartesian product for our graphs. In order to maintain the readability of our example, we introduce some notation. The arrows, $a \in S$, will be denoted in the following way: (1) first-order arrows from vertex x to vertex y will be denoted by $[xy]$; second-order arrows from $[uv]$ to $[xy]$ will be denoted by $[\![[uv][xy]]\!]$; and third-order order from $[wz]$ to $[\![[uv][xy]]\!]$ as $[\![[wz][\![[uv][xy]]\!]]\!]$.

Product of FSGs

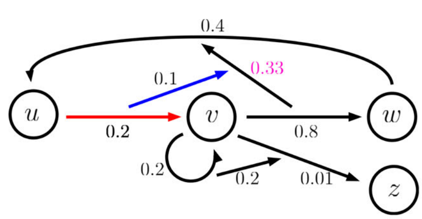
Definition 16 Given two FSGs with disjoint sets of states: $M = \langle W, \mu : S \to [0, 1]\rangle$ and $N = \langle V, \kappa : T \to [0, 1]\rangle$, the **Cartesian product** of M and N is the FSG: $M \times N = \langle W \times V, \psi : (W \times T) \cup (S \times V) \to [0, 1]\rangle$, s.t. $\psi(w, t) = \kappa(t)$ and $\psi(s, v) = \mu(s)$.

Example 11 Let be the FSGs, M and N, in Fig. 7. Figure 8 shows the product $M \times N$.

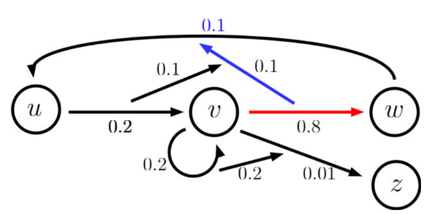
Observe that the membership function, $\psi : (W \times T) \cup (S \times V) \to [0, 1]$, captures the fuzzy values in Fig. 8. First-order arrows like: $(a, e) \overset{0.7}{\to} (a, f)$ and $(c, f) \overset{0.8}{\to} (b, f)$ are represented by $\psi(a, [ef]) = 0.7$ and $\psi([cb], f) = 0.8$, respectively.

Second-order arrows like: $[(a, e)(a, f)] \overset{0.3}{\to} [(a, f)(a, e)]$ and $[(c, f)(b, f)] \overset{0.4}{\to} [(a, f)(b, f)]$ are represented by $\psi(a, [\![[ef][fe]]\!]) = 0.3$ and $\psi([\![[cb][ab]]\!], f) = 0.4$.

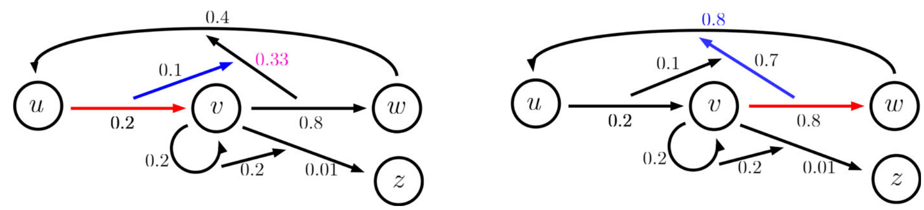
Fig. 5 Reactivity after crossing first-order arrows



(a) First step: Configuration $M^{(u,v,0.2)}_{arith}$ obtained after crossing (u, v).



(b) Second step: Configuration $(M^{(u,v,0.2)}_{arith})^{(v,w,0.8)}_{arith}$ obtained after crossing (v, w).



(a) $M^{a_1^0}$. **(b)** $M^{a_3^0}$.

Fig. 6 Reactivity in Fig. 3 after crossing first-order arrows a_1^0 or a_3^0, which activates the aggregations "*arith*" and "max," respectively

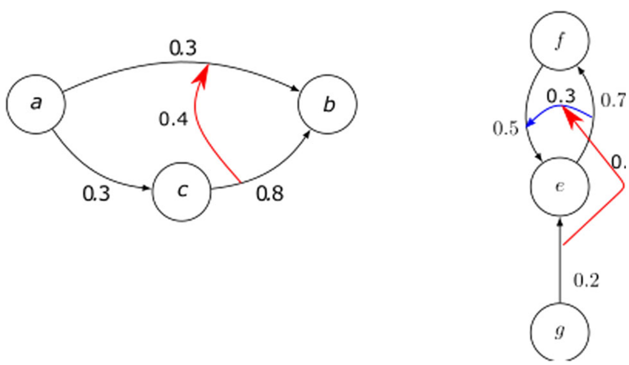


Fig. 7 Fuzzy switch graphs M and N, respectively

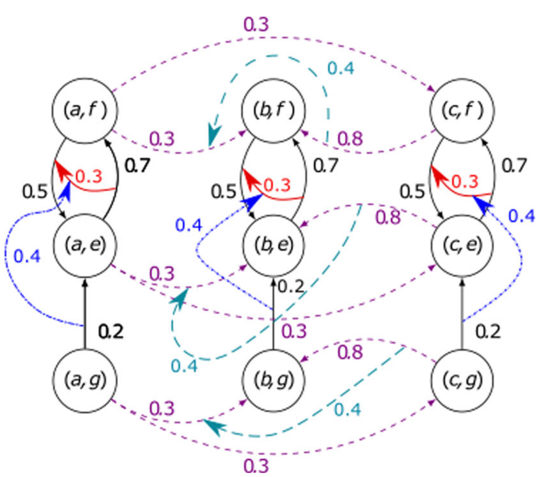


Fig. 8 Product of M and N

Third-order arrows like: $[(a,g)(a,e)] \overset{0.4}{\rightarrow} [\![(a,e)(a,f)]$ $[(a,f)(a,e)]\!]$ are represented as $\psi(a, [ge][\![[ef][fe]]\!]) = 0.4$. Therefore, the graph in Fig. 8 can be functionally described in the following way. For all $w \in W$ and $v \in V$,

- **Firs-order arrows:** $\psi(w, [ef]) = 0.7$, $\psi(w, [fe]) = 0.5$, , $\psi(w, [ge]) = 0.2$, $\psi([ac], v) = \psi([ab], v) = 0.3$ and $\psi([cb], v) = 0.8$.
- **Second-order arrows:** $\psi(w, [\![[ef][fe]]\!]) = 0.3$ and $\psi([\![[cb][ab], v]\!]) = 0.4$.
- **Third-order arrows:** $\psi(w, [ge][\![[ef][fe]]\!]) = 0.4$.

Product of FRGs

Definition 17 Given two FRGs with disjoint sets of states $\langle M, Ag_M \rangle$ and $\langle N, Ag_N \rangle$ the aggregation functions s.t. $Ag_M : S_M^0 \rightarrow A_M$ and $Ag_N : S_N^0 \rightarrow A_N$, $a_m \in A_M$ and $a_n \in A_N$. Let be the aggregation functions $(M, a_m), (N, a_n) : [0,1]^3 \rightarrow [0,1]$ s.t. $(M, a_m)(x_1, x_2, x_3) = a_m(x_1, x_2, x_3)$ and $(N, a_n)(x_1, x_2, x_3) = a_n(x_1, x_2, x_3)$. Let be the set $A_M \oplus A_N = \{(M, a_m) : a_m \in A_M\} \cup \{(N, a_n) : a_n \in A_N\}$, the set of first-order arrows $\Gamma_{M\times N} = \{a_i^0 : a_i^0 \in (W \times T^0) \cup (S^0 \times V)\}$ and the function $Ag_{M\times N} : \Gamma_{M\times N} \rightarrow A_M \oplus A_N$ s.t:

$$Ag_{M\times N}(a_i^0) = \begin{cases} (N, Ag_N(t)), & \text{if } a_i^0 = (w,t) \in W \times T^0 \\ (M, Ag_M(s)), & \text{if } a_i^0 = (s,v) \in S^0 \times V. \end{cases} \tag{6}$$

The structure $\langle M \times N, Ag_{M\times N} \rangle$ is the **product of FRGs** M and N. For simplicity, we also denote this product by: $M \times N$ (assuming that M and N are FRGs).

Proposition 4 *Given the product of two FRGs M and N, $M \times N$, then:*

$$\psi_{a_i^0}^{Ag_{M\times N}}(a) = \begin{cases} \kappa(t), & \text{if } C_1; \\ \mu(s), & \text{if } C_2; \\ Ag_N(t)\Big(\psi(a_i^0), \psi(a_i^0, a), \psi(a)\Big), & \text{if } C_3; \\ Ag_M(s)\Big(\psi(a_i^0), \psi(a_i^0, a), \psi(a)\Big), & \text{if } C_4. \end{cases} \tag{7}$$

For $C_1 : a = (w,t)$ and $(a_i^0, a) \notin (W \times T) \cup (S \times V)$; $C_2 : a = (s,v)$ and $(a_i^0, a) \notin (W \times T) \cup (S \times V)$; $C_3 : a = (w,t)$ and $(a_i^0, a) \in (W \times T) \cup (S \times V)$; and $C_4 : a = (s,v)$ and $(a_i^0, a) \in (W \times T) \cup (S \times V)$.

Proof – Case $(a_i^0, a) \notin (W \times T) \cup (S \times V)$,

- ∗ Case $a = (w,t)$, $\psi_{a_i^0}^{Ag_{M\times N}}(a) \stackrel{\text{def}}{=} \psi(a) \stackrel{\text{def}}{=} \kappa(t)$.
- ∗ Case $a = (s,v)$, $\psi_{a_i^0}^{Ag_{M\times N}}(a) \stackrel{\text{def}}{=} \psi(a) \stackrel{\text{def}}{=} \mu(s)$.

– Case $(a_i^0, a) \in (W \times T) \cup (S \times V)$,

* Case $a = (w, t)$, $\psi_{a_i^0}^{Ag_{M\times N}}(a) \stackrel{\textbf{def}}{=} Ag_{M\times N}(a_i^0)\Big(\psi(a_i^0), \psi(a_i^0, a), \psi(a)\Big) = \Big(N, Ag_N(t)\Big)\Big(\psi(a_i^0), \psi(a_i^0, a), \psi(a)\Big) \stackrel{\textbf{def}}{=} Ag_N(t)\Big(\psi(a_i^0), \psi(a_i^0, a), \psi(a)\Big)$.
* Case $a = (s, v)$, $\psi_{a_i^0}^{Ag_{M\times N}}(a) \stackrel{\textbf{def}}{=} Ag_{M\times N}(a_i^0)\Big(\psi(a_i^0), \psi(a_i^0, a), \psi(a)\Big) = \Big(M, Ag_M(s)\Big)\Big(\psi(a_i^0), \psi(a_i^0, a), \psi(a)\Big) \stackrel{\textbf{def}}{=} Ag_M(s)\Big(\psi(a_i^0), \psi(a_i^0, a), \psi(a)\Big)$.

□

Notes 1 Although we have introduced FRG in general, during the rest of the document, for simplicity, we will consider FRGs with a single aggregation A and we write A instead of the unique mapping $Ag : S^0 \to \{A\}$.

4 Connection between FRGs and fuzzy graphs

A fuzzy reactive graph (FRG) $\Big(\langle W, \mu : S \to [0, 1]\rangle, A\Big)$, can be transformed in an equivalent fuzzy graph (with no higher-level arrows).

Definition 18 Given a fuzzy reactive graph, $\Big(\langle W, \mu : S \to [0, 1]\rangle, A\Big)$, let be the **family of admissible fuzzy subsets of** S, Ω—which is the least set containing μ and such that $\bar{\mu} \in \Omega$ whenever $\bar{\mu} = \mu'^{A}_{(w,w')}$ for some $\mu' \in \Omega$ and $(w, w') \in S_0$ such that $\mu'(w, w') > 0$. Consider $\tilde{W} = \{(w, \mu) \in W \times \Omega\}$ and $\tilde{R} = \tilde{W} \times \tilde{W} \to [0, 1]$ such that

$$\tilde{R}\big((w, \mu), (w', \mu')\big) = \begin{cases} \mu(w, w'), & \text{if } \mu' = \mu^{A}_{(w,w')} \\ 0, & \text{otherwise} \end{cases}$$

The fuzzy graph $\langle \tilde{W}, \tilde{R}\rangle$ is called the **fuzzy graph induced by** $\mathcal{M}$.

Remark 1 The next example is very important, since it illustrates how, for some cases, infinite fuzzy graphs can be represented by equivalent finite fuzzy switch graphs. In this way, it is possible to obtain a finite representation for a infinite fuzzy graph. Indeed, this can occur due to the choice of some specific aggregations functions, as shown in Fig. 9.

Example 12 Consider the FSG M in Fig. 7, the second projection, $A_l(x, y, z) = y$, the arithmetic mean, $A_r = (x, y, z) = \dfrac{x + y + z}{3}$, and the FRGs (M, A_l) and (M, A_r). The respective induced fuzzy graphs of (M, A_l) and (M, A_r) are in Fig. 9.

Note that the induced fuzzy graph remains finite for the second projection whereas becomes infinite for the arithmetic mean. This process of reducing infinite fuzzy graph to finite FRG is expected to be studied in future work with detail.

5 A Logic for fuzzy switch graphs

Now we have FSGs to model the system configurations, how do we verify that the modeled system has or does not have a certain property? Further, is it possible to have a calculus for that? In what follows, we show how we can use logic with FSGs.

First, assume that a state u is interpreted as the *initial state* of a system and a state w is the *state of success*. Whenever the state z is reached, it means that the system almost *fails*. The state v is where the system almost always *reloads and tries again.*

Now, suppose we have uncertainty degrees relating some atomic propositions (properties) and the states of the system. For example, take the atomic propositions: *The system succeed, the system fails,* and *the system is trying* according to the following table:
meaning, for example, that the "degree of uncertainty that the system *is trying at state u* is 0.1". The truth value of compound propositions: Conjunctions, disjunctions, conditionals, negations and bi-conditionals can be calculated by using: T-norms, T-conorms, fuzzy negations, implications and bi-implications, respectively.

Moreover, on each state, w, it is also possible to verify the uncertainty degree of propositions like:

> "***At some*** *next state, s, the proposition φ holds*" as well as "***In all the*** *next state, s, the proposition φ holds*."

In order to show how logic can be used to verify properties of a FSG, we provide a formal language and a fuzzy semantics.

5.1 Language

Syntax

Given a set of symbols: $AtomProp$, called set of atomic propositions, the set of formulas is defined by the following grammar:

$\varphi ::= p \mid true \mid false \mid (\neg\varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid (\varphi \leftrightarrow \varphi) \mid (\mathbb{S}_{\mathtt{Next}}(\varphi)) \mid (\mathbb{A}_{\mathtt{Next}}(\varphi))$, for $p \in AtomProp$. [1]

A formula that only contains the operators $\vee$, $\wedge$ and `S_next` is called *positive formula*. The formulas φ will be read in the following way:

[1] For clarity, we will omit the external parenthesis whenever it is possible, e.g. we will write $\varphi \wedge \psi$ instead of $(\varphi \wedge \psi)$.

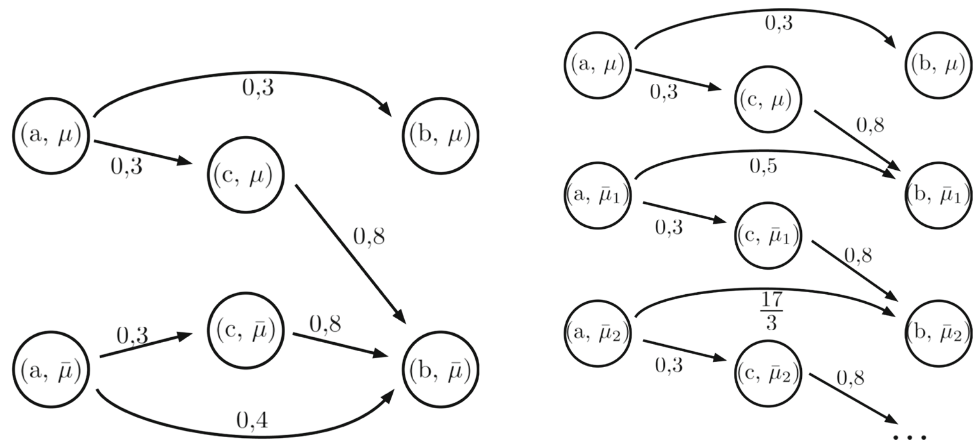


Fig. 9 Fuzzy graphs obtained from (M, A_l) and (M, A_r)

Table 3 Truth values of propositions on each state

	u	v	w	z
succ	0.2	0.8	0.3	0.01
fail	0.7	0	0.3	0.99
try	0.1	0.9	0.15	0.2

$(\neg\varphi)$	means	φ is not true
$(\varphi \wedge \psi)$	means	φ and ψ are true
$(\varphi \vee \psi)$	means	φ or ψ is true
$(\varphi \rightarrow \psi)$	means	If φ is true, then ψ is true
$(\varphi \leftrightarrow \psi)$	means	φ is true if and only if ψ is true
$(\mathbb{S}_{\texttt{Next}}(\varphi))$	means	φ is true in some next state
$(\mathbb{A}_{\texttt{Next}}(\varphi))$	means	φ is true in all next states.

The formal interpretation of such formulas will be given in Definition 21.

5.2 Semantics

Definition 19 A **model** is a pair, $\mathcal{M} = (M, V)$, where $V : W \times AtomProp \rightarrow [0, 1]$ is a function called **fuzzy valuation** and M is a FSG.

Definition 20 Given a model $\mathcal{M} = (M, V)$ and N a subgraph of $M = (W, \mu)$, we say that $\mathcal{N} = (N, \tilde{V})$ is a **submodel** of $\mathcal{M} = (M, V)$ whenever, $\tilde{V}(w, p) \leq V(w, p)$ for every state $w \in W$ and $p \in AtomProp$.

A model enables the users to assign the grade of uncertainty that each atomic proposition holds at each state w. Table 3 shows an example of a fuzzy evaluation V. Now, starting from a model, the following definition provides an algorithm which calculates the grade of uncertainty that an *arbitrary* formula, φ, have at a state w:

Definition 21 Given a model $\mathcal{M} = (M, V)$, an aggregation function A and a fuzzy semantics: $\mathcal{F} = \langle [0, 1], T, S, N, I, B, 0, 1\rangle$, the **grade of uncertainty of a given formula**, φ, **at state** $w \in W$, taking into account $\mathcal{M}$, $\mathcal{F}$ and A, denoted by

$$[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!],$$

is recursively calculated in the following way:

- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} p]\!] = V(w, p)$, for $p \in AtomProp$.
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} true]\!] = 1$.
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} false]\!] = 0$.
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \wedge \psi]\!] = \mathbf{T}\big([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!]\big)$.
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \vee \psi]\!] = \mathbf{S}\big([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!]\big)$.
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \rightarrow \psi]\!] = \mathbf{I}\big([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!]\big)$.
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \leftrightarrow \psi]\!] = \mathbf{B}\big([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!]\big)$.
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \neg\varphi]\!] = \mathbf{N}\big([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!]\big)$.
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \mathbb{A}_{\texttt{Next}}(\varphi)]\!] = \underset{w' \in S^0[w]}{\mathbf{T}}\Big(\mathbf{I}\Big(\mu(w, w'), [\![\mathcal{M}_A^{(w,w',\mu(w,w'))}, w' \vDash^A_{\mathcal{F}} \varphi]\!]\Big)\Big)$; where $\mathcal{M}_A^{(w,w',\mu(w,w'))} = (M_A^{(w,w',\mu(w,w'))}, V)$ and $S^0[w] = \{w' \in W : (w, w') \in S^0\}$.[2]
- $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \mathbb{S}_{\texttt{Next}}(\varphi)]\!] = \underset{w' \in S^0[w]}{\mathbf{S}}\Big(\mathbf{T}\Big(\mu(w, w'), [\![\mathcal{M}_A^{(w,w',\mu(w,w'))}, w' \vDash^A_{\mathcal{F}} \varphi]\!]\Big)\Big)$.[3]

---

[2] First, observe that $w' \in S^0[w]$ is equivalent to $(w, w') \in S^0$. The notation $\underset{w' \in S^0[w]}{\mathbf{T}}\Big(I\Big(\mu(w, w'), [\![M^{(w,w',\mu(w,w')}, w' \vDash \tfrac{A}{F}\phi]\!]\Big)\Big)$, in short $\underset{w' \in S^0[w]}{\mathbf{T}}(f(w'))$ means the iterative application of a T-norm — which is a binary operation — on $f(w') = I\Big(\mu(w, w'), [\![M^{(w,w',\mu(w,w')}, w' \vDash \tfrac{A}{F}\varphi)]\!]\Big)$, for $w' \in S^0[w]$. That is, for $S^0[w] = \emptyset$, $\underset{w' \in S^0[w]}{\mathbf{T}}(f(w')) = 1$; for $S^0[w] = \{v\}$, $\underset{w' \in S^0[w]}{\mathbf{T}}(f(w')) = f(v)$; for $S^0[w] = \{v_1, v_2\}$, $\underset{w' \in S^0[w]}{\mathbf{T}}(f(w')) = T(f(v_1), f(v_2))$; for $S^0[w] = \{v_1, v_2, v_3\}$, $\underset{w' \in S^0[w]}{\mathbf{T}}(f(w')) = T(f(v_1), T(f(v_2), f(v_3)))$ and so on. Note that there is no ambiguity in this notation, since a T-norm is commutative and associative we do not need to consider an order on $S^0[w]$.

[3] See the previous footnote.

Note that this definition generalizes the usual one for Fuzzy models (Jain et al. 2020) (based on Fuzzy graphs, with no higher-level edges). Hence, we write $[\![\mathcal{M}, w' \vDash_{\mathcal{F}} \varphi]\!]$, since aggregations are not considered for fuzzy graphs.

We must highlight, contrary to what happens in the classic case, the truth of $\mathtt{S}_{\mathtt{Next}}(\varphi)$ and $\mathtt{A}_{\mathtt{Next}}(\varphi)$ at a state w deals with all edges in S^0 with source w. The expression: $|[\mathcal{M}_A^{(w,w',\mu(w,w'))}, w' \vDash_{\mathcal{F}}^{A} \varphi|]$, in this case, represents the uncertainty degree that: "φ holds" at state w' after the first-order edge: $a_i^0 = (w, w', \mu(w, w'))$ has been crossed and the FSG M has been updated to M_A. In what follows, we provide an example of how it does work.

Once the user has modeled the reactivity of its system, by providing a FSG, the verification that his/her system has some property can be performed in the following way:

Example 13 (Calculus by using the Gödel Semantics) Now, assuming the values in Table 3, what is the uncertainty at state v that: "I*n some next state we have a fail and there is a next state in which the system succeeds*"? The assertion can be expressed in our formal language as:

$$\mathtt{S}_{\mathtt{Next}}(\mathtt{S}_{\mathtt{Next}}(succ) \wedge fail) \tag{8}$$

The grade of uncertainty that the statement (8) holds at some state of a model will depend on the adopted aggregation for the underlying FSG and the fuzzy semantics in which the statement is interpreted. For example, based on the aggregation adopted in Fig. 5: The arithmetical mean, and assuming the previous T-norms, T-conorms, fuzzy negations, implications and bi-implications. If (8) is interpreted in the structure:

$$\mathcal{F}^G = \langle [0, 1], T_M, S_M, N_G, I_G, B_G, 0, 1 \rangle,$$

then (as we will see) we obtain 0.2 as the uncertainty grade that (8) holds. This grade will change whenever another fuzzy semantics is provided. In what follows, we present the notion of fuzzy semantics and show how we can compute the uncertainty grade for a statement like (8).

Consider the Gödel semantics, $\mathcal{F}^G$, and the arithmetic mean as the aggregation A. Over the FSG in Fig. 3, which we call M, consider the three atomic propositions: $succ$, $faill$ and try, previously described, with evaluation given in Table 3. In what follows, we show how to calculate the uncertainty degree for two propositions holds in the state v. The first formula is the previous (8), and the second is: "$\mathtt{A}_{\mathtt{Next}}(succ \vee \neg fail)$."

1. Making $\varphi = \mathtt{S}_{\mathtt{Next}}(succ) \wedge fail$, we have: $[\![\mathcal{M}, v \vDash_{\mathcal{F}}^{A} \mathtt{S}_{\mathtt{Next}}(\varphi)]\!]$ $\overset{\mathbf{def}}{=\!=} S_M\Big[T_M\Big(0.2, [\![\mathcal{M}_A^{(v,v,0.2)}, v \vDash_{\mathcal{F}}^{A} \varphi]\!]\Big),$ $T_M\Big(0.01, [\![\mathcal{M}_A^{(v,z,0.01)}, z \vDash_{\mathcal{F}}^{A} \varphi]\!]\Big),$ $T_M\Big(0.8, [\![\mathcal{M}_A^{(v,w,0.8)}, w \vDash_{\mathcal{F}}^{A} \varphi]\!]\Big)\Big] = 0.2$. In fact,

(a)

$$\min(0.2, \min([\![\mathcal{M}_A^{(v,v,0.2)}, v \vDash_{\mathcal{F}}^{A} \mathtt{S}_{\mathtt{Next}}(succ)]\!], [\![\mathcal{M}_A^{(v,v,0.2)}, v \vDash_{\mathcal{F}}^{A} fail]\!]) = \min\big(0.2, \min([\![\mathcal{M}_A^{(v,v,0.2)}, v \vDash_{\mathcal{F}}^{A} \mathtt{S}_{\mathtt{Next}}(succ)]\!], 0)\big) = 0,$$

(b)

$$\min\big(0.01, \min([\![\mathcal{M}_A^{(v,z,0.01)}, z \vDash_{\mathcal{F}}^{A} \mathtt{S}_{\mathtt{Next}}(succ)]\!], [\![\mathcal{M}_A^{(v,z,0.01)}, z \vDash_{\mathcal{F}}^{A} fail]\!]) = \min(0.01, \min(0, 0.99)) = 0, \text{ and}$$

(c)

$$\min(0.8, \min([\![\mathcal{M}_A^{(v,w,0.8)}, w \vDash_{\mathcal{F}}^{A} \mathtt{S}_{\mathtt{Next}}(succ)]\!], [\![\mathcal{M}_A^{(v,w,0.8)}, w \vDash_{\mathcal{F}}^{A} fail]\!])) = \min\big(0.8, \min([\![\mathcal{M}_A^{(v,w,0.8)}, w \vDash_{\mathcal{F}}^{A} \mathtt{S}_{\mathtt{Next}}(succ)]\!], 0.3)\big) = \min(0.8, \min(\min(\frac{1.9}{3}, (\mathcal{M}_A^{(v,w,0.8)})_A^{(w,u,\frac{1.9}{3})}, u \vDash_{\mathcal{F}}^{A}, succ), 0.3)) = \min(0.2, 0.3) = 0.2.$$

This can be interpreted in the following way:

"The uncertainty degree that *from the state v there is a state x in which: (a) we reach a posterior state such that the system succeeds and (b) the system fails is 0.2"*

2. Similarly, $[\![\mathcal{M}, v \vDash_{\mathcal{F}}^{A} \mathtt{A}_{\mathtt{Next}}(succ \vee \neg fail)]\!] = T_M\Big(I_G\big(0.2, [\![\mathcal{M}_A^{(v,v,0.2)}, v \vDash_{\mathcal{F}}^{A} (succ \vee \neg fail)]\!]\big), I_G\big(0.01, [\![\mathcal{M}_A^{(v,z,0.01)}, z \vDash_{\mathcal{F}}^{A} (succ \vee \neg fail)]\!]\big), I_G\big(0.8, [\![\mathcal{M}_A^{(v,w,0.8)}, w \vDash_{\mathcal{F}}^{A} (succ \vee \neg fail)]\!]\big)\Big) = \min(I_G(0.2, (\max(0.8, 1)), I_G(0.01, (\max(0.01, 0)), I_G(0.8, \max(0.3, 0)))) = \min(1, 1, 0.3) = 0.3$.

We can interpret this as:

"0.3 is the grade of uncertainty that *all next states of v are successful states or states with no failure."*

Proposition 5 *Let $\mathcal{N} = (N, \tilde{V})$ be a submodel of $\mathcal{M} = (M, V)$. Then, $[\![\mathcal{N}, w \vDash_{\mathcal{F}}^{A} \varphi]\!] \leq [\![\mathcal{M}, w \vDash_{\mathcal{F}}^{A} \varphi]\!]$ for every positive formula φ.*

Proof We prove this result by induction over the structure of positive formulas and due to the increasingness of aggregations, S-norms and T-norms.

– It holds for atomic propositions by definition and trivially for *true* and *false*.

–

$$\begin{aligned}&[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \wedge \psi]\!]\\ &= \mathbf{T}([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!])\\ &\geq \mathbf{T}([\![\mathcal{N}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{N}, w \vDash^A_{\mathcal{F}} \psi]\!])\\ &= [\![\mathcal{N}, w \vDash^A_{\mathcal{F}} \varphi \wedge \psi]\!]\end{aligned}$$

–

$$\begin{aligned}&[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \vee \psi]\!]\\ &= \mathbf{S}([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!])\\ &\geq \mathbf{S}([\![\mathcal{N}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{N}, w \vDash^A_{\mathcal{F}} \psi]\!])\\ &= [\![\mathcal{N}, w \vDash^A_{\mathcal{F}} \varphi \vee \psi]\!]\end{aligned}$$

–

$$\begin{aligned}&[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}\, \varphi]\!]\\ &= \underset{w' \in S_0[w]}{\mathbf{S}} \Big(\mathbf{T}\big(\mu_M(w, w'), [\![\mathcal{M}, w' \vDash^A_{\mathcal{F}} \varphi]\!]\big)\Big)\\ &\geq \underset{w' \in S_0[w]}{\mathbf{S}} \Big(\mathbf{T}\big(\mu_N(w, w'), [\![\mathcal{N}, w' \vDash^A_{\mathcal{F}} \varphi]\!]\big)\Big)\\ &= [\![\mathcal{N}, w \vDash^A_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}\, \varphi]\!]\end{aligned}$$

□

Definition 22 Given a FRG $\big(M, A\big)$ and a model $\mathcal{M} = (M, V)$, the pair $\tilde{\mathcal{M}} = \Big((\tilde{W}, \tilde{R}), \tilde{V}\Big)$ with $\tilde{V}\big((w, \mu), p\big) = V(w, p)$ is called **induced fuzzy model of $\mathcal{M}$ by A.**

Theorem 1 *Given a FRG $\big(M, A\big)$ and a fuzzy model, $\mathcal{M} = (M, V)$, then $[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!] = [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi]\!]$.*

Proof We prove this result by induction over the structure of formulas.

– It holds for atomic propositions by definition and trivially for *true* and *false*.

–

$$\begin{aligned}[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \wedge \psi]\!] &= \mathbf{T}([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!])\\ &= \mathbf{T}([\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi]\!], [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \psi]\!])\\ &= [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi \wedge \psi]\!].\end{aligned}$$

–

$$\begin{aligned}[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \vee \psi]\!] &= \mathbf{S}([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!])\\ &= \mathbf{S}([\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi]\!], [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \psi]\!])\\ &= [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi \vee \psi]\!].\end{aligned}$$

–

$$\begin{aligned}[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \to \psi]\!] &= \mathbf{I}([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!])\\ &= \mathbf{I}([\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi]\!], [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \psi]\!])\\ &= [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi \to \psi]\!].\end{aligned}$$

–

$$\begin{aligned}[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi \leftrightarrow \psi]\!] &= \mathbf{B}([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \psi]\!])\\ &= \mathbf{B}([\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi]\!], [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \psi]\!])\\ &= [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi \leftrightarrow \psi]\!].\end{aligned}$$

–

$$\begin{aligned}[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \neg\varphi]\!] &= \mathbf{N}([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \varphi]\!])\\ &= \mathbf{N}([\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \varphi]\!]) = [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \neg\varphi]\!]\end{aligned}$$

–

$$\begin{aligned}&[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}\, \varphi]\!]\\ &= \underset{w' \in S^0[w]}{\mathbf{S}} \Big(\mathbf{T}\big(\mu(w, w'), [\![\mathcal{M}_A^{(w,w',\mu(w,w'))}, w' \vDash^A_{\mathcal{F}} \varphi]\!]\big)\Big)\\ &= \underset{w' \in S^0[w]}{\mathbf{S}} \Big(\mathbf{T}\big(\mu(w, w'), [\![(\widetilde{\mathcal{M}_A^{(w,w',\mu(w,w'))}}),\\ &\quad (w', \mu^A_{(w,w')}) \vDash_{\mathcal{F}} \varphi]\!]\big)\Big), \text{ by I.H.}\\ &= \underset{w' \in S^0[w]}{\mathbf{S}} \Big(\mathbf{T}\big(\mu(w, w'), [\![\tilde{\mathcal{M}}, (w', \mu^A_{(w,w')}) \vDash_{\mathcal{F}} \varphi]\!]\big)\Big),\\ &\quad \text{since } (\widetilde{\mathcal{M}_A^{(w,w',\mu(w,w'))}})\\ &\text{is, by definition, the routed submodel of } \tilde{\mathcal{M}} \text{ at}\\ &\quad (w', \mu^A_{(w,w')})\\ &= [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}\, \varphi]\!], \text{ by definition.}\end{aligned}$$

–

$$\begin{aligned}&[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \mathtt{A}_{\mathtt{Next}}\, \varphi]\!]\\ &= \underset{w' \in S^0[w]}{\mathbf{T}} \Big(\mathbf{I}\big(\mu(w, w'), [\![\mathcal{M}_A^{(w,w',\mu(w,w'))} w' \vDash^A_{\mathcal{F}} \varphi]\!]\big)\Big)\\ &= \underset{w' \in S^0[w]}{\mathbf{T}} \Big(\mathbf{I}\big(\mu(w, w'),\\ &[\![(\widetilde{\mathcal{M}_A^{(w,w',\mu(w,w'))}}), (w', \mu^A_{(w,w')}) \vDash_{\mathcal{F}} \varphi]\!]\big)\Big), \text{ by I.H.}\\ &= \underset{w' \in S^0[w]}{\mathbf{T}} \Big(\mathbf{I}\big(\mu(w, w'),\\ &[\![\tilde{\mathcal{M}}, (w', \mu^A_{(w,w')}) \vDash_{\mathcal{F}} \varphi]\!]\big)\Big), \text{ as in previous}\\ &= [\![\tilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{F}} \mathtt{A}_{\mathtt{Next}}\, \varphi]\!], \text{ by definition.}\end{aligned}$$

□

5.3 Bisimulation

We introduce a notion of bisimulation for FSGs. This definition is based on the one by Jain et al. (2020) for fuzzy graphs (see also Cao et al. 2012).

Notation. Given a relation $E \subseteq W \times W'$ and $w \in W$, $E[w] \overset{\mathbf{def}}{=\!=} \{w' \in W' : (w, w') \in E\}$ and $(w', w) \in E^{-1} \Leftrightarrow (w, w') \in E$.

Definition 23 (Jain et al. 2020) Let $M = \langle W, \mu \rangle$ and $M' = \langle W', \mu' \rangle$ be two fuzzy graphs. A relation $E \subseteq W \times W'$ is said to be a **bisimulation** between the fuzzy models $\mathcal{M} = (M, V)$and $\mathcal{M}' = (M', V')$ if, for every $(w, w') \in E$:

1. $V(w, p) = V'(w', p)$ for every $p \in AtomProp$;
2. for any $u \in W$, $\mu(w, u) \leq \sup_{u' \in E[u]} \mu'(w', u')$;
3. for any $u' \in W'$, $\mu'(w', u') \leq \sup_{u \in E^{-1}[u']} \mu(w, u)$.

In any case, if E verifies 1 and 2, then it is said that E **simulates** M in M'.

Lemma 1 (Jain et al. 2020) *Considering the Gödel semantics, given fuzzy models* $\mathcal{M} = (\langle W, \mu \rangle, V)$ *and* $\mathcal{M}' = (\langle W', \mu' \rangle, V')$*, and a bisimulation* $E \subseteq W \times W'$ *s.t.* $(w, w') \in E$*. Then,* $[\![\mathcal{M}', w' \vDash \varphi]\!] = [\![\mathcal{M}, w \vDash \varphi]\!]$ *for every formula* φ.

Definition 24 Let $M = (\langle W, \mu \rangle, A)$, $M' = (\langle W', \mu' \rangle, A)$ be two FRGs and $E \subseteq W \times W'$. Regarding the induced fuzzy graphs $\tilde{M} = \langle \tilde{W}, \tilde{R} \rangle$ and $\widetilde{M'} = \langle \widetilde{W'}, \tilde{R'} \rangle$, the relation $\tilde{E} \subseteq \tilde{W} \times \widetilde{W'}$ is a *generalization of* E if $((w, \mu), (w', \mu')) \in \tilde{E}$ whenever $(w, w') \in E$.

Definition 25 Given two FRGs M and M', a relation $E \subseteq W \times W'$ is a *bisimulation between* the fuzzy models $\mathcal{M} = \langle M, V \rangle$ and $\mathcal{M}' = \langle M', V' \rangle$, if there is a generalization $\tilde{E}$ which is a bisimulation between $\tilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}'}$.

Theorem 2 *Given two fuzzy reactive graphs* $M = (\langle W, \mu \rangle, A)$, $M' = (\langle W', \mu' \rangle, A)$ *and a bisimulation* $E \subseteq W \times W'$*, containing* (w, w')*, between the fuzzy models* $\mathcal{M} = (M, V)$ *and* $\mathcal{M}' = (M', V)$*. Then,* $[\![\mathcal{M}', w' \vDash^A_{\mathcal{G}} \varphi]\!] = [\![\mathcal{M}, w \vDash^A_{\mathcal{G}} \varphi]\!]$ *for every formula* φ*, with* $\mathcal{G}$ *the Gödel semantics.*

Proof By Definition 25, Lemma 1 and Theorem 1,

$$\begin{aligned} [\![\mathcal{M}', w' \vDash^A_{\mathcal{G}} \varphi]\!] &= [\![\widetilde{\mathcal{M}'}, (w', \mu') \vDash_{\mathcal{G}} \varphi]\!] \\ &= [\![\widetilde{\mathcal{M}}, (w, \mu) \vDash_{\mathcal{G}} \varphi]\!] = [\![\mathcal{M}, w \vDash^A_{\mathcal{G}} \varphi]\!]. \end{aligned}$$ □

6 A biological application

In biology, we can find occurrences of reactive behaviors. For instance, we can mention the case of vaccination and prodrugs as presented in Figueiredo and Barbosa (2019). Another is in the Rhesus incompatibility between the fetus and the mother (Megginson et al. 1996). This incompatibility may occur when the mother is Rhesus negative and the fetus is Rhesus positive. If the mother's blood already contacted with Rhesus positive blood, the immune system of the mother has already developed antibodies against it and her blood will not be compatible with the fetus, since the immune system will attack the baby. If no contact happened before, the pregnancy may expect no complications since no antibodies were created and the blood of the mother is not expected to contact with the blood of the fetus till his birth. In other words, the contact with Rhesus positive alters (or "reconfigures") the mother's immunological system. In what follows, we show how some biological systems can be modeled by FSGs.

6.1 Circadian rhythm in cyanobacteria

In this example, we consider the system of the circadian rhythm of a cyanobacteria as described in Chaves and Preto (2013). In order to explain this example, we present some background about biological models for regulatory networks.

A regulatory networks are composed of a set of components—genes, proteins, mRNA, etc.—along with a relation of regulation (which can reflect either a positive or negative regulation). In this context, we say that a component i positively (respectively, negatively) regulates a component j if the presence of i induces (respectively, inhibits) the production of j. This network of regulations can be studied with several kinds of models, ranging from ordinary differential equations (ODE) models (which are the most descriptive) to Boolean networks (which are graphs only describing the overall dynamics of the system). For more information, see (De Jong 2002).

ODE models consider variables, x_i, relative to each component, i. This variable describes the component concentration within a cell. Then, a positive regulation of i over j is described by the differential equation $x'_j = k_{ji} s^+(x_i) = k_{ji} \dfrac{x_i^n}{x_i^n + \theta_{ji}^n}$ and a negative regulation by $x'_j = k_{ji} s^-(x_i)$ with $s^-(x_i) = 1 - s^+(x_i)$. The thresholds $k_j i$ and $\theta_j i$ are estimated according to the value chosen for $n \in \mathbb{N}$. The complete ODE model has the form $x' = F(x)$ is thus obtained by a system of equations where each equation $x_i = F_i(x)$ is obtained gathering every regulations over i. There, $F_i(x)$ is described as sums and products of functions $s^+(x_j)$ and $s^-(x_j)$, as described before, along with a degradation term $-\gamma_i x_i$.

In Chaves and Preto (2013), the thresholds for a ODE model were estimated and we can thus present the following ODE model for the circadian rhythm system where the considered components are three phosphorilated forms of the KaiC protein—s, t and ts—an unphosphorilated form of KaiC protein—u—and KaiA protein—a.

$$\begin{cases} x_a' = 10\dfrac{5^4}{x_s^4 + 5^4} - 0.45x_a \\ x_t' = 20.51\dfrac{x_u^4}{x_u^4 + 29.95^4}\dfrac{x_a^4}{x_a^4 + 10^4} - 0.24x_t \\ x_{ts}' = 10.74\dfrac{x_t^4}{x_t^4 + 11.42^4}\dfrac{x_a^4}{x_a^4 + 10^4} - 0.28x_{ts} \\ x_s' = 6.61\dfrac{x_{ts}^4}{x_{ts}^4 + 10.16^4}\dfrac{13^4}{x_a^4 + 13^4} - 0.081x_s \end{cases}$$

where $x_u + x_s + x_t + x_{ts} = C$, is a constant value.

The use of this kind of model to study biological regulatory networks has a drawback: *It is difficult to study a system of nonlinear differential equations*. Since, in general, only simulations can be performed, we can consider a simplification of this kind of model by considering $n \to \infty$. In this context, we observe that the expressions representing positive regulations are simplified. For an arbitrary variables x and arbitrary constant θ:

$$\frac{x^n}{x^n + \theta^n} \xrightarrow{n\to\infty} \begin{cases} 1, & \text{if } x > \theta \\ \frac{1}{2}, & \text{if } x = \theta \\ 0, & \text{if } x < \theta \end{cases}$$

In this way, we can think qualitatively about the values of a variable x. Given some thresholds θ_0,...,θ_p: we can divide the entire state space in several domains divided by the conditions $x = \theta_0$, ..., $x = \theta_n$, defining hyperplanes. In this way, within each domain we have a system of linear differential equations, which can now be solved analytically. Each of this region can be seen as a qualitative evaluation for the value of the variables. For instance, if we consider two thresholds θ_0 and θ_1 for a variable x, we can say that the concentration of the respective component is "low" $x < \theta_0$, "medium" if $\theta_0 < x < \theta_1$ and "high" if $\theta_1 < x$.

This kind of model is called PWL (PieceWise Linear) and since the solution of the differential equation within each domain is analytically solvable, given an initial state, we can determine the induced flow (trajectory of the solution). In this way, it is possible to think about transitions between domains and in concept like adjacency: We say that two n-dimensional domains are adjacent if they share a $(n - 1)$-dimensional boundary. We say that it is possible to move from a domain "A" to another one "B" when exists a *flow* guiding us from A to the boundary with an adjacent domain B, *i.e.*, the solution of the system of linear ODEs, with respect to time, gives us a trajectory that leads us from a domain "A" to another domain "B." We note that, formally, sometimes it is not possible to cross the boundaries given the way flows are defined. This occurs when flows have opposite directions in small neighborhoods of the boundary but, we are not interested in these cases and we ignore them for now.

With the concept of transition between domain, we can think of Boolean networks. This kind of model is simply an oriented graph whose set of nodes correspond to the set of domains in the respective PWL model. Moreover, the set of edges is obtained according to the possible moves between domains, given by the flows. In order to recover the general dynamics of the biological system, an order is assigned to the variables and the nodes are labeled by tags like "00," meaning that the value of the first two variables is below the lowest threshold. This order is important for such models because we need to know which value of the node tag represents the qualitative concentration of each component. Classically, tags only contain "0"s – for "low concentration" —and "1"s—for "high concentration"—and this is the reason why this kind of model is called Boolean network. However, the states of such model can be labeled with tag like "23" in the case that more qualitative intervals are chosen for variables. For instance, in the "23" case, "2" may mean "high" and "3" may mean "very high".

Recalling the example of circadian rhythm in cyanobacteria, the Boolean network obtained is as shown in Fig. 10 with each of the variables x_a, x_s, x_{ts} and x_t. The variable x_u was not considered since it can be replaced by $C - (x_s + x_{ts} + x_t)$, for the constant value C introduced before. Also, for the variables representing the concentration of components, we consider three values for x_a—0 for low, 1 for medium and 2 for high—and two values for x_t, x_{ts} and x_s – 0 for low and 1 for high. The order for variables is (x_a, x_t, x_{ts}, x_s) and this means that, for instance in state 0011, the concentration of a and t is low and the concentration of ts and s is high. Also, at state 1111, the concentration of a is medium while the concentration of t, ts, and s is high.

Given one of these models, it is usual to study asymptotic behaviors like steady states of the original ODE model. These steady states correspond to either stable/unstable equilibrium points or limit cycles and they are signaled in Boolean networks by sets of nodes with special properties called attractors. In Figueiredo and Barbosa (2019), a reactive model was already proposed to improve this search for attractors. However, in this paper, our approach follows for another direction with the introduction of fuzziness.

In the Boolean network for the circadian rhythm, edges represent a possible transition between domains of state spaces. However, this does not mean that each transition is equally likely to occur. Indeed, some edges have higher chance of being crossed than others. Taking the state labeled by 1001 in Fig. 10, and studying the respective domain of the corresponding PWL model, we can found that edges coming out from that state have quite different probabilities of being crossed as shown in Fig. 11. The boundary crossed depends on the exact trajectory; however, since the model is simplified and divided by domains, we cannot determine this and only that probabilities can be computed.

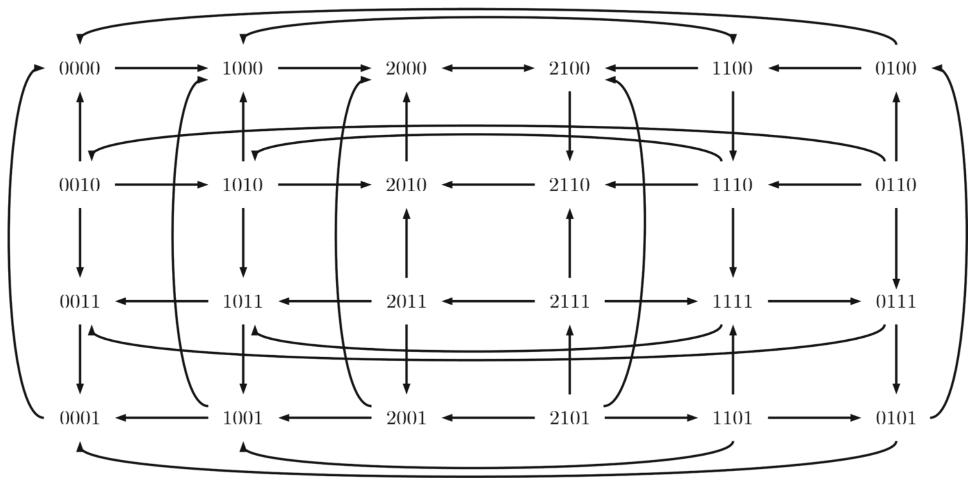


Fig. 10 Boolean network for the circadian rhythm of cyanobacteria

Fig. 11 State 1001 and edges coming out from it

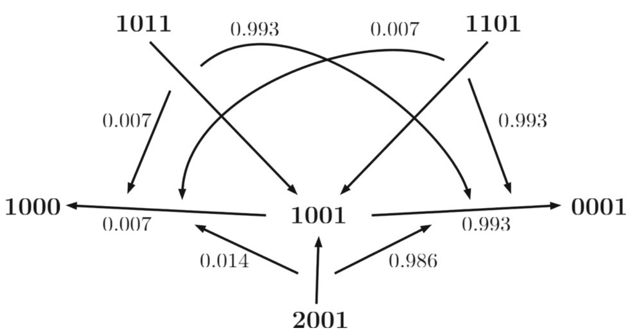


Fig. 12 Fuzzy switch graph of partial Boolean network

At this point, we can think about reactivity. In fact, the probabilities presented in Fig. 11 can vary taking into account the previous states. In fact, when one knows the previous domain can restrict the set of admissible flows to those which come from the desired domain. In Fig. 12, we can construct a fuzzy switch graph for the edges where the state 1001 takes part. In this example, the fuzzy aggregation function considered is $A(a, b, c) = b$. The reactivity in this context makes sense because, provided a set of initial states and restricting the solutions of differential equations in PWL models to the one which only consider those initial states, we can see how the probabilities alter. In these examples, the values for probabilities were obtained numerically from the model.

We note that we only consider part of the fuzzy switch graph in order to illustrate the utility of fuzzy switch graphs. The entire model could be obtained but would be difficult to include its graphical representation in this paper.

Moreover, this example highlight the convenience of using a reactive model. Indeed, they naturally fit in the class of reconfigurable discrete models and, in cases like the one presented in this example, it can be seen as a discrete abstraction of hybrid system (system which comprise both continuous and discrete dynamics). This is, indeed the basis for the work in Figueiredo and Barbosa (2019).

6.2 Cooperativity of hemoglobin

In a weighted context, we can consider a red blood cell at the lungs. Hemoglobin in these cells is responsible for delivering oxygen to cells in the entire body after binding it at the lungs. Each hemoglobin protein gathers a limited number of four molecules of oxygen as further shown in Fig. 13. In the example being presented, an hemoglobin protein collecting oxygen molecules is considered. While initially the protein is relatively "uninterested" on oxygen, the scenario changes after one has been bound and even more when the second and third oxygen molecules are also bind. This process is called cooperativity (see Chou 1989).

Here, a hemoglobin protein is represented. The protein bind up to 4 oxygen molecules at an increasing rate, depending on the number of oxygen that it has already bound. This is illustrated in Fig. 13 where the only node represents the hemoglobin protein and the regular edge represent the action "bind an oxygen molecule." For this example, the suitable aggregation function is defined as $A(a, b, c) = min\left(b, \frac{b+c}{2}\right)$.

Thinking of weights as fuzzy rates, the model indicates that, initially, oxygen molecules are not easily bound, since the value of the first-order arrow is 0.2. However, after it is crossed once, its value increases to 0.6 and, after being successively crossed, to 0.8 and 0.9. Finally, its value becomes 0, *meaning that* it is no more possible for the hemoglobin protein to bind any other oxygen molecule. This is illustrated in Fig. 14.

Next, we provide an example of how to use the previously presented logic to prove properties of the model.

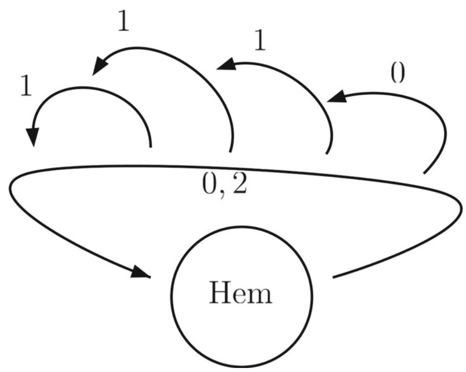


Fig. 13 Fuzzy switch graph modeling a hemoglobin protein

Considering the Gödel Fuzzy semantics $\mathcal{F}^G$ and the FSG, M, presented in Fig. 13 such that w names its only node. We now show that

$$[\![M, w \vDash^A_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}\, true \rightarrow (\mathtt{S}_{\mathtt{Next}}\, (\mathtt{S}_{\mathtt{Next}}\, true))]\!] = 1, \quad (9)$$

what means that *it is true* that the rate increases after the first movement.

$$[\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}\, true \rightarrow (\mathtt{S}_{\mathtt{Next}}(\mathtt{S}_{\mathtt{Next}}\, true))]\!]$$
$$= I_G([\![\mathcal{M}, w \vDash^A_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}\, true]\!],$$
$$[\![\, \mathcal{M}, w \vDash^A_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}(\mathtt{S}_{\mathtt{Next}}\ true)]\!])$$
$$= I_G\Big(\min\big(0.2,\ [\![\mathcal{M}_A^{(w,w,0.2)}, w \vDash^A_{\mathcal{F}} true]\!]\big),$$
$$\min\big(0.2,\ [\![\mathcal{M}_A^{(w,w,0.2)}, w \vDash^A_{\mathcal{F}} \mathtt{S}_{\mathtt{Next}}\, true)]\!]\big)$$
$$= I_G\Big(\min 0.2, 1), \min\big(0.2,\ \min\big(0.6,$$
$$[\![(\mathcal{M}_A^{(w,w,0.2)})_A^{(w,w,0.6)}, w \vDash^A_{\mathcal{F}}\ true]\!])\big)\big)$$
$$= I_G\Big(0.2, \min\big(0.2, \min 0.6, 1)\big)\Big) = I_G(0.2, 0.2) = 1.$$

Similarly, it can also be computationally verified that for any valuation, the following formula holds:

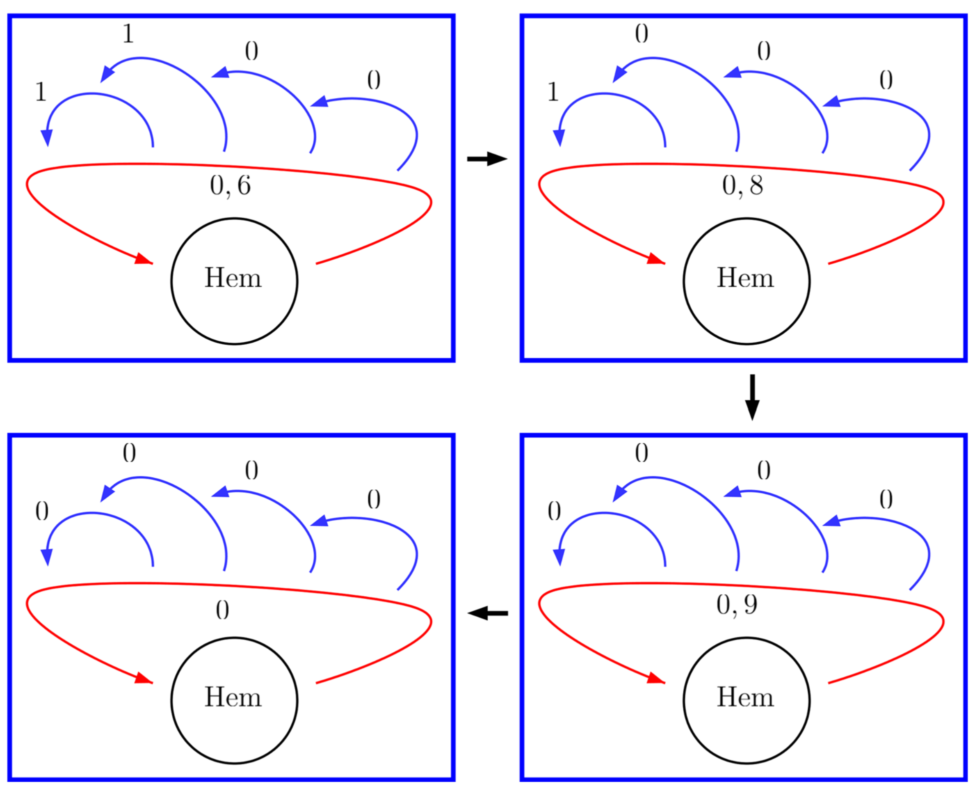
$$\mathcal{M}, w \vDash^A_{\mathcal{F}}$$
$$\Big(\mathtt{S}_{\mathtt{Next}}\, (\mathtt{S}_{\mathtt{Next}}\, (\mathtt{S}_{\mathtt{Next}}\, (\mathtt{S}_{\mathtt{Next}}(\mathtt{A}_{\mathtt{Next}}\ false))))\Big) \quad (10)$$

It means that: *It is impossible to cross the loop more than four times*, proving that the model is correct with respect to the fact that the capture of oxygen molecules occurs, at most, four times.

7 Final remarks

This paper introduces a new fuzzy notion, called *fuzzy reactive graphs* **(FRG)**, which generalizes the concept of fuzzy graphs. The model is accompanied with a calculus which enables the user to verify the grade of uncertainty that some

Fig. 14 Evolution of hemoglobin protein when binding oxygen molecules successively



properties hold in a specific state of the system. As a model, during our work, fuzzy reactive graphs have shown up to be especially appropriated to describe dynamics of systems where the reactive behavior is caused by an unknown cause or component. Note that, although we do not know the direct cause for some of behavior, we are still able to model such system.

We have introduced a connection between FRGs and fuzzy graphs which enabled us to define the notion of bisimulation on FRGs. We point out that it was shown the possibility to obtain finite FRGs which represents the dynamics of infinite fuzzy graphs, which highlights the potential of our structures. Indeed, as future work, some reduction algorithms based on this should be developed. Also, we intend to introduce a definition of bisimulation for FRGs without the corresponding definition for fuzzy models. This definition is expected to match the one for meaningful bisimulations.

We have shown that, in some specific cases, FRGs are suitable to simplify other mathematical models like PWL, used to describe biological systems. A further study providing a comparison between FRGs and other mathematical tools which capture the notions of dynamics and reactivity is the subject of future works.

We point out that our approach is fuzzy (which is not probabilistic). However, another approach using Markov chains and switch graphs can be found in Figueiredo et al. (2019).

Finally, the language presented here could be enriched with many other operators (e.g., by introducing an interation operator) as well as it can be straightforwardly generalized to other fuzzy semantics, i.e., it is parameterized on the choice of the underlying fuzzy algebra. Indeed, this is still an introductory paper but our goal is to introduce a more expressive language for these models as well as a better characterization of its expressiveness. Furthermore, a computational tool to automatically check formulas of this logic is also left as future work.

Finally, we are working on an appropriate direct definition of bisimulation for, i.e., without making reference to the induced fuzzy graphs and on other algebraic construction of FRGs like: Co-products, Union, Intersection, etc.

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Compliance with ethical standards

Conflict of interest The authors declare no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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