Universidade de Aveiro Departamento de Matemática
2019

Pedro Miguel Teixeira Olhero Pessoa Nora

Dualidades de Kleisli e coálgebras de Vietoris

Kleisli dualities and Vietoris coalgebras

Universidade de Aveiro Departamento de Matemática
2019

Pedro Miguel Teixeira Olhero Pessoa Nora

Dualidades de Kleisli e coálgebras de Vietoris

Kleisli dualities and Vietoris coalgebras

Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Prof. Dr. Dirk Hofmann, Professor Associado com agregação do Departamento de Matemática da Universidade de Aveiro.

Dissertation submitted to the University of Aveiro in fulfilment of the requirements for the degree of *Doctor of Philosophy* in Mathematics, under the supervision of Prof. Dr. Dirk Hofmann, Associate Professor with "agregação" of the Department of Mathematics of the University of Aveiro.

Fundação para a Ciência e a Tecnologia

COMPETE 2020

O júri / The jury

Presidente / President	**Doutor Carlos Fernandes da Silva** Professor Catedrático, Universidade de Aveiro (por delegação da Reitoria da Universidade de Aveiro)
Vogais / Examiners Committee	**Doutor João José Neves Silva Xarez** Professor auxiliar, Universidade de Aveiro
	Doutora Maria Manuel Pinto Lopes Ribeiro Clementino Professora Catedrática, Universidade de Coimbra
	Doutora Adriana Balan Leitora, University Politechnica of Bucharest
	Doutor Isar Stubbe Professor Associado, Université du Littoral-Côte D'Opale
	Doutor Dirk Hofmann Professor Associado com agregação, Universidade de Aveiro (Orientador)

Acknowledgements

An unsuspecting reader might think that this is a thesis about mathematics. That's "not even wrong!". It's about friendship. And this . . . this is our diary.

A good conversation is worth a thousand lectures. I am forever indebted to Professor Walter Tholen for the inspiring conversations during my stay at York University.

Partial financial assistance by Portuguese funds through CIDMA (Center for Research and Development in Mathematics and Applications), and the Portuguese Foundation for Science and Technology ("FCT – Fundação para a Ciência e a Tecnologia"), within the project UID/MAT/04106/2013 and the FCT grant SFRH/BD/95757/2013 is gratefully acknowledged.

This work was partially supported by the ERDF – European Regional Development Fund through the Operational Programme for Competitiveness and Internationalisation - COMPETE 2020 Programme and by National Funds through the Portuguese funding agency, FCT - Fundação para a Ciência e a Tecnologia, within project POCI-01-0145-FEDER-030947.

Palavras-chave

Equivalência dual; equivalência; dualidade; categoria de Kleisli; mónada; adjunção; quantale; topologia; espaço estávelmente compacto; espaço de Hausdorff; espaço espectral; espaço Booleano; métrica; ordem; espaço compacto parcialmente ordenado; espaço de Priestley; reticulado distributivo; álgebra Booleana; coalgebra de Vietoris; functor polinomial de Kripke; functor polinomial de Vietoris; limite; limite codireccionado.

Resumo

Nesta tese pretendemos estender de forma sistemática dualidades de Stone-Halmos para categorias que incluem todos os espaços de Hausdorff compactos. Para atingir este objectivo combinamos teoria de dualidades e teoria de categorias enriquecidas em quantais. A nossa ideia principal é que ao passar do espaço discreto com dois elementos para um cogerador da categoria de espaços de Hausdorff compactos, todas as restantes estruturas envolvidas devem ser substituídas por versões enriquecidas correspondentes. Desta forma, consideramos o intervalo unitário $[0, 1]$ e desenvolvemos teoria de dualidades para espaços ordenados compactos e categorias enriquecidas em $[0, 1]$ finitamente cocompletas (apropriadamente definidas). Na segunda parte da tese estudamos limites em categorias de coalgebras cujo functor subjacente é um functor de Vietoris polinomial — intuitivamente, uma versão topológica de um functor polinomial de Kripke.

Keywords

Dual equivalence; duality; Kleisli category; monad; adjunction; enriched category; quantale; topology; stably compact; compat Hausdorff; spectral space; Boolean space; Stone space; metric; order; partially ordered compact; Priestley space; distributive lattice; Boolean algebra; Vietoris coalgebra; Kripke polynomial functor; Vietoris polynomial functor; limit; codirected limit.

Abstract

In this thesis we aim for a systematic way of extending Stone-Halmos duality theorems to categories including all compact Hausdorff spaces. To achieve this goal, we combine duality theory and quantale-enriched category theory. Our main idea is that, when passing from the two-element discrete space to a cogenerator of the category of compact Hausdorff spaces, all other involved structures should be substituted by corresponding enriched versions. Accordingly, we work with the unit interval $[0,1]$ and present duality theory for ordered compact spaces and (suitably defined) finitely cocomplete categories enriched in $[0,1]$. In the second part, we study limits in categories of coalgebras whose underlying functor is a Vietoris polynomial one — intuitively, the topological analogue of a Kripke polynomial functor.

Contents

Chapter 1

Introduction

Matisse, the French artist famous for using bright and expressive colours, once said: “I don’t paint things. I *only* paint the differences between things.”. It is quite likely that this quote feels familiar to a mathematician, but also a bit odd! After all, by painting (all) differences we end up emphasising what is equal. For a category theorist, “emphasising equality” might be perceived as the process of “painting equivalences” of categories. Of course, just like for Matisse, it is the (apparent) perception of difference that makes painting exciting; in a similar way that discovering an equation like $e^{i\omega} = \cos(\omega) + i\sin(\omega)$ is far more interesting than realising that $3 = 3$. Numerous examples of interesting equivalences of seemingly different categories relate a category X and the dual of a category A. Such an equivalence is called a dual equivalence or simply a duality, and is usually denoted by $\mathsf{X} \simeq \mathsf{A}^{\mathrm{op}}$. Like every other equivalence, a duality allows us to transport properties from one side to the other. The presence of the dual category is often useful because our knowledge about a category is typically asymmetric. Indeed, many “everyday categories” admit a representable and, therefore, limit preserving functor to Set. In these categories limits are typically “easy”, however, colimits are often “hard”. Then, an equivalence $\mathsf{X} \simeq \mathsf{A}^{\mathrm{op}}$ together with the knowledge of limits in A help us understand colimits in X. The dual situation, where colimits are “easy” and limits are “hard”, occurs frequently in the context of coalgebras.

The prime example of coalgebras motivating this thesis are coalgebras for the Vietoris functor V on the categoy BooSp of Boolean spaces[1] and continuous maps as investigated in [Kupke et al., 2004]. In this case, it is easy to study limits by “changing the perspective”. It is well-know that the category $\mathrm{CoAlg}(V)$ of coalgebras for this Vietoris functor is equivalent to the dual of the category BAO with objects Boolean algebras B with an operator $h\colon B \to B$ satisfying the equations

$$h(\bot) = \bot \qquad \text{and} \qquad h(x \vee y) = h(x) \vee h(y),$$

[1] Also called Stone spaces in the literature.

and with morphisms the Boolean homomorphisms which also preserve the additional unary operation. It is a trivial observation that BAO is a category of algebras over Set defined by a (finite) set of operations and a collection of equations, and every such category is known to be complete and cocomplete. Remarkably, the equivalence $\mathrm{CoAlg}(V) \simeq \mathsf{BAO}^{\mathrm{op}}$ allows to conclude the non-trivial fact that $\mathrm{CoAlg}(V)$ is complete. This argument also shows that, starting with a category X, the category A in a dual equivalence $\mathsf{X} \simeq \mathsf{A}^{\mathrm{op}}$ does not need to be a familiar category. It is certainly beneficial that $\mathsf{A} = \mathsf{BAO}$ is a well-known category, however, every algebraic category describable by a set of operations would be sufficient to conclude completeness of $\mathsf{X} = \mathrm{CoAlg}(V)$.

This example lies exactly at the intersection of the main topics of this thesis. The Vietoris functor on BooSp is part of a monad $\mathbb{V}$, and, as advocated by Halmos in [Halmos, 1956], the duality above as well as the classical Stone duality for Boolean algebras are consequences of a general duality involving the Kleisli category of $\mathbb{V}$.

Theorem (Halmos' dual equivalence)**.** *The Kleisli category* $\mathsf{BooSp}_{\mathbb{V}}$ *of the Vietoris monad on* BooSp *is dually equivalent to the category* $\mathsf{FinSup}_{\mathrm{BA}}$ *of Boolean algebras and finite suprema preserving maps.*

Halmos gives a direct proof for this result. He does not, however, talk about Kleisli categories or even about monads; instead, he refers to Boolean relations which happen to correspond precisely to morphisms in $\mathsf{BooSp}_{\mathbb{V}}$ as shown in Kupke et al. [2004]. This observation allows to tackle Halmos duality indirectly with the help of monad theory, and marks the beginning of our journey into duality theory.

The main goal of this thesis is to illustrate how to combine monad theory and quantale-enriched category theory to unify arguments and dualities involving Kleisli categories. More concretely, we are looking to extend Halmos' dual equivalence to categories including all compact Hausdorff spaces in a way that the objects of the corresponding dual category appear as generalisations of Boolean algebras. In the case of Halmos' duality, our approach highlights the role of the two-element discrete space as a cogenerator in the category BooSp. To pass to the category $\mathsf{CompHaus}$ of compact Hausdorff spaces and continuous maps for example, we would need to replace the two-element discrete space with a cogenerator of $\mathsf{CompHaus}$ such as the unit interval. Together with the Vietoris functor on $\mathsf{CompHaus}$, this idea by itself could lead us to a Halmos version of Gelfand's duality theorem (see [Gelfand, 1941]).

Theorem (Gelfand's dual equivalence)**.** *The category* $\mathsf{CompHaus}$ *is dually equivalent to the category* C^*-Alg *of* C^**-algebras and homomorphisms.*

But to pass from functions to continuous *relations*, what part of the structure of a C^*-algebra the morphisms need to ignore? The answer is not obvious, and even if it were, at best we would end up with a duality result where the objects of the dual category of $\mathsf{CompHaus}$ do not seem generalisations of Boolean algebras. To improve upon this, we resort to quantale-enriched category theory. Our thesis is that *the passage from the two-element space to the compact Hausdorff space* $[0, 1]$ *should be matched on the algebraic side of Halmos' duality by a move from ordered structures (*2*-categories) to metric structures (*$[0, 1]$*-categories).*

Roughly speaking, in analogy with the results for the two-element space, we are looking for an equivalence functor (or at least a full embedding)

$$\mathsf{CompHaus} \longrightarrow (\text{metric spaces with some (co)completeness properties})^{\mathrm{op}}$$

and, more generally, with $\mathsf{StablyComp}$ denoting the category of stably compact spaces and spectral maps, a full embedding

$$\mathsf{StablyComp} \longrightarrow (\text{metric spaces with some (co)completeness properties})^{\mathrm{op}},$$

that follow from a general result about a full embedding of the Kleisli category $\mathsf{StablyComp}_{\mathbb{V}}$ of the Vietoris monad $\mathbb{V}$ on $\mathsf{StablyComp}$:

$$\mathsf{StablyComp}_{\mathbb{V}} \longrightarrow (\text{“finitely cocomplete” metric spaces})^{\mathrm{op}}.$$

The notion of “finitely cocomplete metric space” should be considered as the metric counterpart to semi-lattice, and “metric space with some (co)completeness properties” as the metric counterpart to (distributive) lattice. Getting back to coalgebras, analogously to the case of Halmos' dual equivalence, the duality results that we are looking for should also give us information on the dual of the category of coalgebras of a Vietoris functor on $\mathsf{StablyComp}$. Nevertheless, “changing the perspective” does not make everything easier all the time.

In the second part of this thesis, we deepen our understanding of limits in categories of coalgebras over topological spaces using somewhat more direct methods. A systematic study of limits is a natural research line in the context of coalgebras. Remarkbly, terminal coalgebras encode a notion of canonical behaviour for all coalgebras, and equalisers provide a notion of subystem which is essential to characterise systems induced by coequations.

Now, we are interested not only in coalgebras for the Vietoris functor V but more generally for functors that are “polynomial” in V. Intuitively, these functors — called Vietoris polynomial functors — are topological analogues of Kripke polynomial functors. It turns out that a great deal of this part is devoted to the study of preservation of codirected limits by variations of Vietoris functors. In particular, we show that the categories of (suitably defined) Vietoris coalgebras over categories of stably compact spaces are complete. Moreover, we conclude that categories of Vietoris coalgebras for all topological spaces have equalisers, (certain) codirected limits and, under some conditions, a terminal object. This part of the thesis was developed in collaboration with Renato Neves (HASLab, University of Minho). For an application of these results in the context of *hybrid programs*, see [Neves, 2018].

1.1 Roadmap

After this introduction, in Chapter 2 we briefly review the core concepts and results that we shall need in Chapters 3 and 4. We begin in Section 2.1 by explaining how to prove equivalences of categories using well-known (large) adjunctions that link monads and adjunctions over a fixed category. Section 2.2 surveys the necessary material of quantale-enriched category theory. Regarding topology, in this thesis we are mainly interested in stably compact spaces. In 2.3 we collect some properties about them and show that the epimorphisms and regular monomorphisms in the category of stably compact spaces and spectral maps are the surjective maps and the subspace embeddings, respectively. Then, in 2.4 we introduce the lower and the compact Vietoris monads on the category of all topological spaces and continuous maps, and explain how they are related when restricted to some categories of stably compact spaces. Finally, in 2.5 we combine some well-known results and standard arguments to design the strategy employed later in Section 4.2 to prove the existence of limits in categories of Vietoris coalgebras.

The heart of this thesis is Chapter 3. In 3.1 we discuss how to use the results of 2.1 to deduce in a uniform way duality theorems involving Kleisli categories. We apply the resulting ideas in 3.2. Guided by quantale-enriched category theory, we develop duality theory for the Kleisli category of the Vietoris monad on the category of stably compact spaces and spectral maps. In 3.2.1 we show that this category is embedded in $\aleph_1$-ary quasivariety of $[0,1]$-categories. Then, in 3.2.2 we adapt the classical Stone–Weierstraß theorem to $[0,1]$-categories to obtain dual equivalences.

We conclude this thesis by studying properties of categories of Vietoris coalgebras in Chapter 4. In 4.1 we prove that the dual of the category of coalgebras for the Vietoris functor on the category of stably compact spaces is a $\aleph_1$-ary quasivariety. In 4.2 we study limits in categories of Vietoris coalgebras. In particular, we show that categories of Vietoris coalgebras defined on several types of spaces are complete. Finally, we observe that categories of Vietoris coalgebras over topological spaces and continuous maps have equalisers, (certain) codirected limits and, under some conditions, a terminal object.

Chapter 2

Background

The aim of this chapter is to bring into notice certain concepts and ideas that occur repeatedly throughout this work. Most of the material presented here is nicely handled in the literature, so the exposition style will be that of a summary.

2.1 Kleisli adjunctions

It is known since [Huber, 1961] that one can turn adjunctions into monads by composing the adjoint functors. On the other hand, one can turn monads into adjunctions in several ways, the two extreme constructions are described in [Kleisli, 1965] and [Eilenberg and Moore, 1965]. An obvious question is whether the involved constructions by themselves give rise to (large) adjunctions. The answer is positive, as explained in [Pumplün, 1970, 1988] and [Tholen, 1974]. A less known fact is that by identifying the *fixed* objects of these adjunctions one obtains general principles to prove the equivalence of categories. In this section we explore this idea that will be used extensively in Chapter 3. We begin with arguably the most prosperous concept of category theory. In the words of Mac Lane "adjoint functors arise everywhere" [MacLane, 1971].

Definition 2.1.1. An ***adjunction***

$$\mathsf{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{A} \quad (F \dashv G)$$

consists of

- the left adjoint functor $F\colon \mathsf{X} \to \mathsf{A}$,
- the right adjoint functor $G\colon \mathsf{A} \to \mathsf{X}$,
- the unit natural transformation $\eta\colon 1_{\mathsf{X}} \to GF$, and

- the co-unit natural transformation $\varepsilon\colon FG \to 1_{\mathsf{A}}$

such that the diagrams

commute. We denote additionally an adjoint situation by $F \dashv G$ and $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ according to the necessary level of detail.

For adjunctions $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ and $(F' \dashv G', \eta', \varepsilon')\colon \mathsf{X} \rightleftarrows \mathsf{A}'$ over a fixed base category X, a ***right morphism of adjunctions*** is a functor $J\colon \mathsf{A} \to \mathsf{A}'$ with $G = G'J$, and a ***left morphism of adjunctions*** is a functor $J\colon \mathsf{A} \to \mathsf{A}'$ with $F' = JF$.

We write

$$\mathsf{RAdj}(\mathsf{X})$$

to denote the category of adjunctions and right morphisms of adjunctions over X, and

$$\mathsf{LAdj}(\mathsf{X})$$

to denote the category of adjunctions and left morphisms of adjunctions over X.

Remark 2.1.2. Note that we do *not* require $F' = JF$ in the definition of a right morphism of adjunctions; still, there is a canonical natural transformation $\kappa\colon F' \to JF$ defined as the composite

$$F' \xrightarrow{F'\eta} F'GF = F'G'JF \xrightarrow{\varepsilon'_{JF}} JF.$$

Similarly, for a left morphism of adjunctions J we have a canonical natural transformation $\iota\colon G \to G'J$ defined as the composite

$$G \xrightarrow{\eta'_G} G'F'G = G'JFG \xrightarrow{G'J\varepsilon} G'J.$$

As we will see next, adjunctions and monads are closely related.

Definition 2.1.3. A ***monad*** $\mathbb{T} = (T, m, e)$ on a category X consists of a functor $T\colon \mathsf{X} \to \mathsf{X}$ together with natural transformations $e\colon 1_{\mathsf{X}} \to T$ (unit) and $m\colon TT \to T$ (multiplication) such that the diagrams

commute. For monads $\mathbb{T}, \mathbb{T}'$ on a category X, a ***monad morphism*** $j\colon \mathbb{T} \to \mathbb{T}'$ is a natural transformation $j\colon T \to T'$ such that the diagrams

$$\begin{array}{ccc} & 1 & \\ {}^{e}\swarrow & & \searrow^{e'} \\ T & \xrightarrow[j]{} & T' \end{array} \qquad\qquad \begin{array}{ccc} TT & \xrightarrow{j^2} & T'T' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ T & \xrightarrow[j]{} & T' \end{array}$$

commute, where $j^2 = j_{T'} \cdot Tj = T'j \cdot j_T$.

The category of monads on X and monad morphisms is denoted by

$$\mathsf{Mon}(\mathsf{X}).$$

Examples 2.1.4. 1. The ***identity monad*** $\mathbb{1} = (1, 1, 1)$. Trivially, the identity functor $1\colon \mathsf{X} \to \mathsf{X}$ together with the identity transformation $1\colon 1 \to 1$ forms a monad. It is the initial monad: for every monad $\mathbb{T} = (T, m, e)$ on X, the unit e is the unique monad morphism $\mathbb{1} \to \mathbb{T}$.

2. The ***powerset monad*** $\mathbb{P} = (P, m, e)$ on the category Set of sets and functions. The powerset functor $P\colon \mathsf{Set} \to \mathsf{Set}$ sends each set to its powerset PX and each function $f\colon X \to Y$ to the direct image function $Pf\colon PX \to PY, A \mapsto f[A]$. The X-component of the natural transformation e respectively m is given by "taking singletons" $e_X\colon X \to PX, x \mapsto \{x\}$ and union $m_X\colon PPX \to PX, \mathcal{A} \mapsto \bigcup \mathcal{A}$.

3. The ***filter monad*** $\mathbb{F} = (F, e, m)$ on Set. The filter functor $F\colon \mathsf{Set} \to \mathsf{Set}$ sends a set X to the set FX of all filters on X and, for $f\colon X \to Y$, the map Ff sends a filter $\mathfrak{f}$ on X to the filter $\{B \subseteq Y \mid f^{-1}[B] \in \mathfrak{f}\}$ on Y. The natural transformations $e\colon 1 \to F$ and $m\colon FF \to F$ are given by

$$e_X(x) = \dot{x} = \{A \subseteq X \mid x \in A\} \qquad \text{and} \qquad m_X(\mathfrak{F}) = \{A \subseteq X \mid A^{\#} \in \mathfrak{F}\},$$

for all sets X, $\mathfrak{F} \in FFX$ and $x \in X$, where $A^{\#} = \{\mathfrak{f} \in FX \mid A \in \mathfrak{f}\}$.

4. The ***ultrafilter monad*** $\mathbb{U} = (U, e, m)$ on Set is the submonad of the filter monad that maps a set X to the subset of FX of all ultrafilters of X.

5. The ***filter monad*** $\mathbb{F} = (F_\tau, e, m)$ on the category Top of topological spaces and continuous maps. For a topological space X, $F_\tau X$ is the set of all filters on the lattice of opens of X, equipped with the topology generated by the sets $U^{\#}$, for $U \subseteq X$ open. The continuous map $F_\tau f\colon F_\tau X \to F_\tau Y$, for $f\colon X \to Y$ in Top, and the unit and the multiplication are defined as above. For more information see [Escardó, 1997].

In Section 2.4 we will describe topological counterparts of the powerset monad.

In the remainder of this section we will construct adjunctions

$$\mathsf{RAdj}(\mathsf{X}) \underset{\perp}{\rightleftarrows} \mathsf{Mon}(\mathsf{X})^{\mathrm{op}} \qquad \text{and} \qquad \mathsf{Mon}(\mathsf{X}) \underset{\perp}{\rightleftarrows} \mathsf{LAdj}(\mathsf{X})$$

and identify their fixed objects.

Remark 2.1.5. An adjunction $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ induces an equivalence between the (possibly empty) full subcategories

$$\mathrm{Fix}(\mathsf{X}) = \{X \in \mathsf{X} \mid \eta_X \text{ is an isomorphism}\} \quad \text{and} \quad \mathrm{Fix}(\mathsf{A}) = \{A \in \mathsf{A} \mid \varepsilon_A \text{ is an isomorphism}\}.$$

For details, see [Porst and Tholen, 1991].

We explain now how these results can be used to establish equivalences of categories.

Every adjunction $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ induces a monad $\mathbb{T} = (T, m, e)$ on X defined by

$$T = GF, \qquad e = \eta \qquad \text{and} \qquad m = G\varepsilon_F.$$

Furthermore, every right morphism J of adjunctions $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ and $(F' \dashv G', \eta', \varepsilon')\colon \mathsf{X} \rightleftarrows \mathsf{A}'$ induces a monad morphism

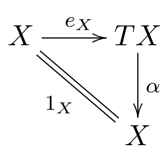
$$j = G'\kappa\colon \mathbb{T}' \to \mathbb{T}$$

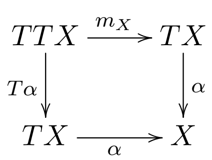
between the induced monads. These constructions define the object and the morphism part of the functor $M^{\mathsf{X}}\colon \mathsf{RAdj}(\mathsf{X}) \to \mathsf{Mon}(\mathsf{X})^{\mathrm{op}}$. Likewise, every left morphism J of adjunctions induces a monad morphism

$$j = \iota_F\colon \mathbb{T} \to \mathbb{T}'$$

and we obtain a functor $M_{\mathsf{X}}\colon \mathsf{LAdj}(\mathsf{X}) \to \mathsf{Mon}(\mathsf{X})$. As we show next, both functors have adjoints given by well-known constructions.

Definition 2.1.6. Let $\mathbb{T} = (T, m, e)$ be a monad on a category X. A $\mathbb{T}$***-algebra*** or a ***Eilenberg-Moore algebra*** for $\mathbb{T}$ is a pair (X, α) consisting of an X-object X and an X-morphism $\alpha\colon TX \to X$ making the diagrams





commute. Let (X, α) and (Y, β) be $\mathbb{T}$-algebras. A map $f\colon X \to Y$ is a $\mathbb{T}$***-algebra homo-***

morphism if the diagram

$$\begin{CD} TX @>{Tf}>> TY \\ @V{\alpha}VV @VV{\beta}V \\ X @>>{f}> Y \end{CD}$$

commutes.

The category of $\mathbb{T}$-algebras and $\mathbb{T}$-algebra homomorphisms is denoted by $\mathsf{X}^{\mathbb{T}}$. There is a canonical forgetful functor $G^{\mathbb{T}}\colon \mathsf{X}^{\mathbb{T}} \to \mathsf{X}, (X,\alpha) \mapsto X$ with left adjoint $F^{\mathbb{T}}\colon \mathsf{X} \to \mathsf{X}^{\mathbb{T}}, X \mapsto (TX, m_X)$. Moreover, every monad morphism $j\colon \mathbb{T} \to \mathbb{T}'$ induces a functor

$$\mathsf{X}^{j}\colon \mathsf{X}^{\mathbb{T}'} \to \mathsf{X}^{\mathbb{T}}, (X,\alpha) \mapsto (X, \alpha \cdot j_X)$$

with $G^{\mathbb{T}}\mathsf{X}^{j} = G^{\mathbb{T}'}$, that is, $\mathsf{X}^{j}\colon (F^{\mathbb{T}'} \dashv G^{\mathbb{T}'}) \to (F^{\mathbb{T}} \dashv G^{\mathbb{T}})$ is a right morphism of adjunctions. These constructions define indeed a functor

$$\mathsf{X}^{(-)}\colon \mathsf{Mon}(\mathsf{X})^{\mathrm{op}} \to \mathsf{RAdj}(\mathsf{X}).$$

It is easy to see that $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ induces $\mathbb{T}$, that is, $\mathbb{T} = M^{\mathsf{X}} \cdot \mathsf{X}^{(-)}(\mathbb{T})$. On the other hand, for every adjunction $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ with induced monad $\mathbb{T}$ we have a canonical ***comparison functor*** $K\colon \mathsf{A} \to \mathsf{X}^{\mathbb{T}}$ defined by $K(A) = (GA, G\varepsilon_A)$ and $Kf = Gf$; hence,

$$K\colon ((F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}) \to ((F^{\mathbb{T}} \dashv G^{\mathbb{T}}, \eta, \varepsilon)\colon \mathsf{X}^{\mathbb{T}} \rightleftarrows \mathsf{X})$$

is a right morphism of adjunctions. For a right morphism J of adjunctions $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ and $(F' \dashv G', \eta', \varepsilon')\colon \mathsf{X} \rightleftarrows \mathsf{A}'$, the diagram

$$\begin{CD} \mathsf{A} @>>> \mathsf{X}^{\mathbb{T}} \\ @V{J}VV @VV{\mathsf{X}^{j}}V \\ \mathsf{A}' @>>> \mathsf{X}^{\mathbb{T}'} \end{CD}$$

commutes, that is, the family of all comparison functors defines a natural transformation $1 \to \mathsf{X}^{(-)}M^{\mathsf{X}}$. In fact, this transformation together with the family $(\mathbb{T} = M^{\mathsf{X}}\mathsf{X}^{(-)}(\mathbb{T}))_{\mathbb{T}}$ are the units of the adjunction

$$\mathsf{RAdj}(\mathsf{X}) \underset{\mathsf{X}^{(-)}}{\overset{M^{\mathsf{X}}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Mon}(\mathsf{X})^{\mathrm{op}}.$$

Clearly, $\mathrm{Fix}(\mathsf{Mon}(\mathsf{X})^{\mathrm{op}}) = \mathsf{Mon}(\mathsf{X})^{\mathrm{op}}$; but deviating slightly from Remark 2.1.5, we let the ***fixed*** subcategory $\mathrm{Fix}(\mathsf{RAdj}(\mathsf{X}))$ consist of those objects whose component of the unit is an equivalence of categories, rather than an isomorphism. An object of $\mathrm{Fix}(\mathsf{RAdj}(\mathsf{X}))$ is called a

monadic adjunction, these adjunctions are characterised by the following

Theorem 2.1.7 ([Beck, 1967]). *An adjunction* $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ *is monadic if and only if* G *reflects isomorphisms and* A *has and* G *preserves all* G*-contractible coequaliser pairs.*

Example 2.1.8. The canonical forgetful functor $|-|\colon \mathsf{CompHaus} \to \mathsf{Set}$ from the category of compact Hausdorff spaces and continuous maps has a left adjoint given by Čech–Stone compactification, and it is shown in [Manes, 1969] that this adjunction is monadic. The induced monad on Set is the ultrafilter monad (see item 4 of Examples 2.1.4).

The "equivalence" between monads and monadic adjunctions provides a general principle to prove equivalence of two categories: firstly, show that both categories are part of monadic adjunctions over the same category X; and secondly, show that these adjunctions induce isomorphic monads. This idea was used in [Negrepontis, 1971] to obtain the classical duality theorems of Gelfand and Pontrjagin.

We will see now how $M_{\mathsf{X}}\colon \mathsf{LAdj}(\mathsf{X}) \to \mathsf{Mon}(\mathsf{X})$ is part of an adjunction.

Definition 2.1.9. Let $\mathbb{T} = (T, m, e)$ be a monad over X. The ***Kleisli category*** $\mathsf{X}_{\mathbb{T}}$ has the same objects as X, and a morphism $f\colon X \rightharpoonup Y$ in $\mathsf{X}_{\mathbb{T}}$ is an X-morphism $f\colon X \to TY$. Given morphisms $f\colon X \rightharpoonup Y$ and $g\colon Y \rightharpoonup Z$ in $\mathsf{X}_{\mathbb{T}}$, the composite $g \cdot f\colon X \rightharpoonup Z$ is defined as

$$X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{m_Z} TZ.$$

Then $e_X\colon X \to TX$ is the identity on X in $\mathsf{X}_{\mathbb{T}}$.

We have a canonical adjunction $F_{\mathbb{T}} \dashv G_{\mathbb{T}}\colon \mathsf{X}_{\mathbb{T}} \rightleftarrows \mathsf{X}$, where

$$G_{\mathbb{T}}\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{X},\ \ f\colon X \rightharpoonup Y \mapsto TX \xrightarrow{Tf} TTY \xrightarrow{m_Y} TY$$

and

$$F_{\mathbb{T}}\colon \mathsf{X} \to \mathsf{X}_{\mathbb{T}},\ \ f\colon X \to Y \mapsto X \xrightarrow{f} Y \xrightarrow{e_Y} TY.$$

Example 2.1.10. For the powerset monad $\mathbb{P}$ on Set, the category $\mathsf{Set}_{\mathbb{P}}$ is equivalent to the category Rel of sets and relations by interpreting a map $f\colon X \to PY$ as a relation $X \nrightarrow Y$ from X to Y. Under this equivalence, $F_{\mathbb{P}}\colon \mathsf{Set} \to \mathsf{Set}_{\mathbb{P}}$ corresponds to the inclusion functor $\mathsf{Set} \to \mathsf{Rel}$ and $G_{\mathbb{P}}\colon \mathsf{Set}_{\mathbb{P}} \to \mathsf{Set}$ corresponds to the functor $\mathsf{Rel} \to \mathsf{Set}$ which sends a set X to its powerset PX and a relation $r\colon X \nrightarrow Y$ to the map $PX \to PY$, $A \mapsto r[A]$.

Every monad morphism $j\colon \mathbb{T} \to \mathbb{T}'$ induces a functor $\mathsf{X}_j\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{X}_{\mathbb{T}'}$ which acts as the identity on objects and sends $f\colon X \rightharpoonup Y$ to $X \xrightarrow{f} TY \xrightarrow{j_Y} T'Y$. One clearly has $F_{\mathbb{T}'} = \mathsf{X}_j F_{\mathbb{T}}$, hence X_j is a left morphism of adjunctions and we obtain a functor

$$Kl_{\mathsf{X}}\colon \mathsf{Mon}(\mathsf{X}) \to \mathsf{LAdj}(\mathsf{X}).$$

As before, the induced monad of $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ is $\mathbb{T}$ again, that is, $\mathbb{T} = M_{\mathsf{X}} \cdot Kl_{\mathsf{X}}(\mathbb{T})$. For every adjunction $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ with induced monad $\mathbb{T}$ we have a canonical ***comparison functor*** $C\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}$ sending an object X in $\mathsf{X}_{\mathbb{T}}$ to FX and a morphism $f\colon X \rightharpoonup Y$ to $FX \xrightarrow{Ff} FTY = FGFY \xrightarrow{\varepsilon_{FY}} FY$. Since $F = CF_{\mathbb{T}}$, C is a left morphism of adjunctions. For a left morphism J of adjunctions $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ and $(F' \dashv G', \eta', \varepsilon')\colon \mathsf{X} \rightleftarrows \mathsf{A}'$, the diagram

(2.1.i)
$$\begin{array}{ccc} \mathsf{X}_{\mathbb{T}} & \longrightarrow & \mathsf{A} \\ {\scriptstyle \mathsf{X}_j}\downarrow & & \downarrow{\scriptstyle J} \\ \mathsf{X}_{\mathbb{T}'} & \longrightarrow & \mathsf{A}' \end{array}$$

commutes. Therefore C is the $((F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A})$-component of a natural transformation $Kl_{\mathsf{X}} M_{\mathsf{X}} \to 1$; in fact, this transformation is the co-unit of the adjunction

$$\mathsf{Mon}(\mathsf{X}) \underset{M_{\mathsf{X}}}{\overset{Kl_{\mathsf{X}}}{\rightleftarrows}} \mathsf{LAdj}(\mathsf{X}). \quad (\bot)$$

where the unit $1 \to M_{\mathsf{X}} Kl_{\mathsf{X}}$ is given by $(\mathbb{T} = M_{\mathsf{X}} Kl_{\mathsf{X}}(\mathbb{T}))_{\mathbb{T}}$. By recalling that the comparison functor $C\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}$ is always fully faithful, we obtain an indirect method of proving that a functor is fully faithful.

Theorem 2.1.11. *Let J be a left morphism of adjunctions. If the monad morphism induced by J is an isomorphism then J is fully faithful.*

We call an adjunction a ***Kleisli adjunction*** whenever C is an equivalence. Unlike the situation for monadic adjunctions, Kleisli adjunctions can be easily characterised.

Theorem 2.1.12. *An adjunction $F \dashv G$ is a Kleisli adjunction if and only if F is essentially surjective on objects.*

As for monadic adjunctions, (2.1.i) gives a simple scheme to obtain an equivalence between categories A and A':

Theorem 2.1.13. *A functor $J\colon \mathsf{A} \to \mathsf{A}'$ between Kleisli adjunctions $(F \dashv G, \eta, \varepsilon)\colon \mathsf{X} \rightleftarrows \mathsf{A}$ and $(F' \dashv G', \eta', \varepsilon')\colon \mathsf{X} \rightleftarrows \mathsf{A}'$ is an equivalence provided that $F' = JF$ and the morphism $M_{\mathsf{X}}(J)$ between the induced monads is a natural isomorphism.*

We will illustrate this principle in Section 3.1 and apply it in Section 3.2 to develop duality theory for categories involving compact Hausdorff spaces.

2.2 Quantale-enriched categories

Lawvere showed in Lawvere [1973] that ordered sets and metric spaces share the same underlying structure: they are quantale-enriched categories. This striking observation implies that we can translate and unify arguments between order theory and metric theory if we can "parameterise" them in terms of appropriate quantales. This idea is at the core of the duality results presented in Chapter 3. In this section we survey the basic theory behind quantale-enriched categories. All material presented here is well-known, most of it is covered in the classical sources [Eilenberg and Kelly, 1966], [Lawvere, 1973] and [Kelly, 1982]. An extensive presentation of this theory in the quantaloid-enriched case is in [Stubbe, 2005, 2006, 2007]. In [Kelly and Lack, 2000], [Kelly and Schmitt, 2005] and [Clementino and Hofmann, 2009] are studied certain colimits in enriched categories. Finally, quantale-enriched categories are particular examples of $(\mathbb{T}, \mathcal{V})$-algebras as introduced in [Clementino and Tholen, 2003; Clementino and Hofmann, 2003].

Definition 2.2.1. A commutative and unital ***quantale*** $\mathcal{V}$ is a complete lattice which carries the structure of a commutative monoid $\otimes\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ with unit element $k\in\mathcal{V}$ such that, for every $u\in\mathcal{V}$, $u\otimes -\colon \mathcal{V}\to\mathcal{V}$ preserves suprema.

Therefore, every monotone map $u\otimes -\colon \mathcal{V}\to\mathcal{V}$ has a right adjoint $\hom(u,-)\colon \mathcal{V}\to\mathcal{V}$ that is characterised by

$$u\otimes v\le w \iff v\le \hom(u,w),$$

for all $v,w\in\mathcal{V}$.

Remark 2.2.2. A quantale is a commutative monoid in the monoidal category Sup of complete lattices and suprema preserving maps.

Definition 2.2.3. Let $\mathcal{V}$ be a quantale. A ***$\mathcal{V}$-category*** is a pair (X,a) consisting of a set X and a map $a\colon X\times X\to\mathcal{V}$ that for all $x,y,z\in X$ satisfies the inequalities

$$k\le a(x,x) \qquad\text{and}\qquad a(x,y)\otimes a(y,z)\le a(x,z).$$

Given $\mathcal{V}$-categories (X,a) and (Y,b), a ***$\mathcal{V}$-functor*** $f\colon (X,a)\to(Y,b)$ is a map $f\colon X\to Y$ such that, for all $x,y\in X$,

$$a(x,y)\le b(f(x),f(y)),$$

For every $\mathcal{V}$-category (X,a), $a^\circ(x,y)=a(y,x)$ defines another $\mathcal{V}$-category structure on X, and the $\mathcal{V}$-category $(X,a)^{\mathrm{op}} := (X,a^\circ)$ is called the ***dual*** of (X,a). Clearly, $\mathcal{V}$-categories and $\mathcal{V}$-functors define a category, denoted as $\mathcal{V}$-Cat. The category $\mathcal{V}$-Cat is complete and cocomplete, and the canonical forgetful functor $\mathcal{V}$-$\mathsf{Cat}\to\mathsf{Set}$ preserves limits and colimits. The quantale $\mathcal{V}$ becomes a $\mathcal{V}$-category when equipped with structure $\hom\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V}$. In fact, for every set S, we can form the S-power $\mathcal{V}^S$ of $\mathcal{V}$ which has as underlying set all functions $h\colon S \to \mathcal{V}$ and with $\mathcal{V}$-category structure $[-,-]$ given by

$$[h,l] = \bigwedge_{s\in S} \hom(h(s), l(s)),$$

for all $h, k\colon S \to \mathcal{V}$.

Examples 2.2.4. Our principal examples are the following.

1. The two-element Boolean algebra $2 = \{0,1\}$ of truth values with $\otimes$ given by "and" &. Then $\hom(u,v) = (u \implies v)$ is implication. A 2-category is an ***ordered set***, that is, a set equipped with a reflexive and transitive relation.The category $\mathsf{2\text{-}Cat}$ is equivalent to the category Ord of ordered sets and monotone maps.

2. The complete lattice $[0,\infty]$ ordered by the "greater or equal" relation $\geqslant$ with multiplication $\otimes = +$. Note that the infimum of two numbers is their maximum and the supremum of $S \subseteq [0,\infty]$ is $\inf S$, In this case we have

$$\hom(u,v) = v \ominus u\colon = \max(v-u, 0).$$

   For this quantale, a $[0,\infty]$-category is a generalised metric space à la Lawvere and a $[0,\infty]$-functor is a non-expansive map (see Lawvere [1973]). We denote this category by Met.

3. Of particular interest to us is the complete lattice $[0,1]$ with the usual "less or equal" relation $\leq$, which is isomorphic to $[0,\infty]$ via the map $[0,1] \to [0,\infty]$, $u \mapsto -\ln(u)$ where $-\ln(0) = \infty$. As the examples below show, metric, ultrametric and bounded metric spaces appear as categories enriched in a quantale based on this lattice. In more detail, we consider the following quantale operations on $[0,1]$ with neutral element 1.

   (a) The tensor $\otimes = *$ is the multiplication and then

$$\hom(u,v) = v \oslash u\colon = \begin{cases} \min(\frac{v}{u}, 1) & \text{if } u \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

   Via the isomorphism $[0,1] \simeq [0,\infty]$, this quantale is isomorphic to the quantale $[0,\infty]$ described above, hence $[0,1]\text{-}\mathsf{Cat} \simeq \mathsf{Met}$.

   (b) The tensor $\otimes = \wedge$ is infimum and then

$$\hom(u,v) = \begin{cases} 1 & \text{if } u \leq v, \\ v & \text{otherwise.} \end{cases}$$

In this case, the isomorphism $[0,1] \simeq [0,\infty]$ establishes an equivalence between $[0,1]$-Cat and the category UMet of ultrametric spaces and non-expansive maps.

(c) The tensor $\otimes = \odot$ is the ***Łukasiewicz tensor*** given by $u \odot v = \max(0, u+v-1)$, here $\hom(u,v) = \min(1, 1-u+v) = 1-\max(0, u-v)$. Via the lattice isomorphism $[0,1] \to [0,1]$, $u \mapsto 1-u$, this quantale is isomorphic to the quantale $[0,1]$ with "greater or equal" relation $\geqslant$ and tensor $u \otimes v = \min(1, u+v)$ truncated addition. This observation identifies $[0,1]$-Cat as the category BMet of bounded-by-1 metric spaces and non-expansive maps.

Every $\mathcal{V}$-category (X,a) carries a natural order defined by

$$x \leq y \text{ whenever } k \leq a(x,y),$$

which is extended pointwise to $\mathcal{V}$-functors making $\mathcal{V}$-Cat a *2-category*. Therefore we can talk about adjoint $\mathcal{V}$-functors; as usual, $f\colon (X,a) \to (Y,b)$ is left adjoint to $g\colon (Y,b) \to (X,a)$, written as $f \dashv g$, whenever $1_X \leq gf$ and $fg \leq 1_Y$. Equivalently, $f \dashv g$ if and only if

$$b(f(x),y) = a(x,g(y)),$$

for all $x \in X$ and $y \in Y$. Note that maps f and g between $\mathcal{V}$-categories satisfying the equation above are automatically $\mathcal{V}$-functors.

The natural order of $\mathcal{V}$-categories defines a faithful functor $\mathcal{V}$-$\mathsf{Cat} \to \mathsf{Ord}$. A $\mathcal{V}$-category is called ***separated*** whenever its underlying ordered set is anti-symmetric, and we denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sep}}$ the full subcategory of $\mathcal{V}$-Cat defined by all separated $\mathcal{V}$-categories. Tautologically, an ordered set is separated if and only if it is anti-symmetric. Hence,

$$\mathsf{Ord}_{\mathrm{sep}}$$

denotes the category of all separated ordered sets and monotone maps. In the sequel we will frequently consider separated $\mathcal{V}$-categories to guarantee that adjoints are unique. We note that the underlying order of the $\mathcal{V}$-category $\mathcal{V}$ is just the order of the quantale $\mathcal{V}$, and the order of $\mathcal{V}^S$ is calculated pointwise. In particular, $\mathcal{V}^S$ is separated.

Definition 2.2.5. A $\mathcal{V}$-category (X,a) is called ***$\mathcal{V}$-copowered*** if for every $x \in X$, the $\mathcal{V}$-functor $a(x,-)\colon (X,a) \to (\mathcal{V},\hom)$ has a left adjoint $x \otimes -\colon (\mathcal{V},\hom) \to (X,a)$ in $\mathcal{V}$-Cat.

This operation is better known under the name "$\mathcal{V}$-tensored"; however, we will use the designation "$\mathcal{V}$-copowered" since it is a special case of a *co*limit. Elementwise, this means that for all $x \in X$ and $u \in \mathcal{V}$, there is an element $x \otimes u \in X$, called the u-copower of x, such that for every $y \in Y$

$$a(x \otimes u, y) = \hom(u, a(x,y)).$$

Example 2.2.6. The $\mathcal{V}$-category $\mathcal{V}$ is $\mathcal{V}$-copowered, with copowers given by the multiplication of the quantale $\mathcal{V}$. More generally, for every set S, the $\mathcal{V}$-category $\mathcal{V}^S$ is $\mathcal{V}$-copowered: for every $h \in \mathcal{V}^S$ and $u \in \mathcal{V}$, the u-copower of h is the map $h \otimes u$ defined by $x \mapsto h(x) \otimes u$.

Remark 2.2.7. If (X, a) is a $\mathcal{V}$-copowered $\mathcal{V}$-category with bottom element $\perp$, then, for every $x \in X$ we have

$$a(x \otimes \perp, y) = \hom(\perp, a(x, y)) = k$$

for every $y \in X$. In particular, this means that $x \otimes \perp$ is a bottom element of the $\mathcal{V}$-category (X, a).

Every $\mathcal{V}$-copowered and separated $\mathcal{V}$-category comes equipped with an action $\otimes \colon X \times \mathcal{V} \to X$ of the quantale $\mathcal{V}$ satisfying

1. $x \otimes k = x$,
2. $(x \otimes u) \otimes v = x \otimes (u \otimes v)$,
3. $x \otimes \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \otimes u_i)$;

for all $x \in X$ and $u, v, u_i \in \mathcal{V}$. Conversely, given a separated ordered set X with an action $\otimes \colon X \times \mathcal{V} \to X$ satisfying the three conditions above, one defines a map $a \colon X \times X \to \mathcal{V}$ using the adjunction $x \otimes - \dashv a(x, -)$, for all $x \in X$. It is easy to see that (X, a) is a $\mathcal{V}$-copowered $\mathcal{V}$-category whose order is the order of X and with copowers given by the action of X.

The construction above yields an isomorphism between the category $\mathcal{V}\text{-}\mathsf{CoPow}_{\text{sep}}$ of $\mathcal{V}$-copowered and separated $\mathcal{V}$-categories and copower-preserving $\mathcal{V}$-functors and the category $\mathsf{Ord}^{\mathcal{V}}_{\text{sep}}$ of separated ordered sets equipped with an action from $\mathcal{V}$ satisfying the three conditions above and action-preserving monotone maps

$$\mathcal{V}\text{-}\mathsf{CoPow}_{\text{sep}} \simeq \mathsf{Ord}^{\mathcal{V}}_{\text{sep}}.$$

Remark 2.2.8. The identification of certain metric spaces as ordered sets equipped with an action of $[0, 1]$ allows to spot the appearance of metric spaces where it does not seem obvious at first sight. For instance, [Banaschewski, 1983] studies the functor that maps a compact Hausdorff space X to its lattice DX of continuous functions into $[0, 1]$, and maps a continuous function f to the lattice homomorphism "composition with f". In [Banaschewski, 1983, Proposition 2] it is shown that a lattice homomorphism is in the image of D if and only if it preserves constant functions. And eventually Banaschewski obtains a duality result for compact Hausdorff spaces by considering the algebraic theory of distributive lattices augmented by constants, one for each element of $[0, 1]$. Motivated by the considerations in this section, instead of adding constants we will consider DX as a lattice equipped with the action of $[0, 1]$ defined by

$$(f \otimes u)(x) = f(x) \wedge u,$$

Then [Banaschewski, 1983, Proposition 2] tells us that the lattice homomorphisms that are in the image of D are precisely the action-preserving ones. This way, we interpret Banaschewski's result in terms of $[0,1]$-copowered ultrametric spaces.

The notion of copower in a $\mathcal{V}$-category (X,a) is a special case of the notion of weighted colimit in (X,a), which we recall next. In the remainder of this section, we denote by G the $\mathcal{V}$-category $(1,k)$. Note that G is a generator in $\mathcal{V}$-Cat.

For a quantale $\mathcal{V}$ and sets X, Y, a ***$\mathcal{V}$-relation*** from X to Y is a map $X \times Y \to \mathcal{V}$ and it will be represented by $X \to\!\!\!\!\!\!\mapsto Y$. As for ordinary relations, $\mathcal{V}$-relations can be composed via "matrix multiplication". That is, for $r\colon X \to\!\!\!\!\!\!\mapsto Y$ and $s\colon Y \to\!\!\!\!\!\!\mapsto Z$, the composite $s\cdot r\colon X \to\!\!\!\!\!\!\mapsto Z$ is calculated pointwise by

$$(s\cdot r)(x,z) = \bigvee_{y\in Y} r(x,y)\otimes s(y,z),$$

for every $x\in X$ and $z\in Z$. Note that the structure of a $\mathcal{V}$-category is by definition a reflexive and transitive $\mathcal{V}$-relation, since the axioms dictate that, for a $\mathcal{V}$-category (X,a), $1_X \leq a$ and $a\cdot a \leq a$. A $\mathcal{V}$-relation $r\colon X \to\!\!\!\!\!\!\mapsto Y$ between $\mathcal{V}$-categories (X,a) and (Y,b) is called a ***$\mathcal{V}$-distributor*** (called bimodule in [Lawvere, 1973]) if $r\cdot a \leq r$ and $b\cdot r\leq r$, and we write $r\colon (X,a) \circ\!\!\!\!\to (Y,b)$.

A ***weighted colimit diagram*** in X consists of a $\mathcal{V}$-functor $h\colon A\to X$ and a $\mathcal{V}$-distributor $\psi\colon A \circ\!\!\!\!\to G$ called the ***weight*** of the diagram. A ***colimit*** of such diagram is an element $x_0\in X$ that for every $x\in X$ satisfies the equality

$$a(x_0,x) = \bigwedge_{z\in A} \hom(\psi(z), a(h(z),x)).$$

Colimits of weighted colimit diagrams are unique up to equivalence. A $\mathcal{V}$-functor $f\colon X\to Y$ between $\mathcal{V}$-categories ***preserves*** a weighted colimit x_0 whenever $f(x_0)$ is a colimit of the weighted colimit diagram in Y determined by $fh\colon A\to Y$ and $\psi\colon A \circ\!\!\!\!\to G$ (for more details see [Kelly, 1982]).

Examples 2.2.9.

1. For $A=G$, a weighted colimit diagram in X consists of an element $x\colon G\to X$ and an element $u\colon G \circ\!\!\!\!\to G$ in $\mathcal{V}$; a colimit of this diagram is the u-copower $x\otimes u$ of x.

2. For a family $h\colon I\to X$, $i\mapsto x_i$ in X take the distributor $\psi\colon I \circ\!\!\!\!\to G$ defined by $\psi(z)=k$, for all $z\in I$. Then $\bar{x}$ is a colimit of this diagram precisely when

$$a(\bar{x},x) = \bigwedge_{i\in I} a(x_i,x),$$

for all $x\in X$; that is, $\bar{x}$ is an order-theoretic supremum of $(x_i)_{i\in I}$ preserved by every

$a(-,x)\colon X \to \mathcal{V}^{\mathrm{op}}$. Such supremum is called ***conical supremum***.

Recall that a $\mathcal{V}$-copowered $\mathcal{V}$-category (X,a) can be interpreted as an ordered set equipped with an action from $\mathcal{V}$. In terms of this structure, (X,a) has all conical suprema of a given shape I if and only if every family $(x_i)_{i\in I}$ has a supremum in the ordered set X and, moreover, for every $u \in \mathcal{V}$

$$\left(\bigvee_{i\in I} x_i\right) \otimes u \simeq \bigvee_{i\in I}(x_i \otimes u).$$

This follows from the facts that $\bigvee_{i\in I} x_i \otimes -$ is left adjoint to $a(\bigvee_{i\in I} x_i, -)$ and

$$\mathcal{V} \xrightarrow{\Delta_{\mathcal{V}}} \mathcal{V}^I \xrightarrow{\prod_{i\in I}(x_i\otimes -)} X^I \xrightarrow{\bigvee} X$$

is left adjoint to

$$X \xrightarrow{\Delta_X} X^I \xrightarrow{\prod_{i\in I} a(x_i,-)} \mathcal{V}^I \xrightarrow{\bigwedge} \mathcal{V}.$$

A $\mathcal{V}$-category is called ***cocomplete*** if every weighted colimit diagram admits a colimit. One can show that a $\mathcal{V}$-category is cocomplete if and only if has the two types of colimits described above; in this case the colimit x_0 of an arbitrary diagram $(h\colon A \to X, \psi\colon A \nrightarrow G)$ is calculated as

$$x_0 = \bigvee_{z\in A} h(z) \otimes \psi(z),$$

since

$$a(\bigvee_{z\in A} h(z)\otimes\psi(z), x) = \bigwedge_{z\in A} a(h(z)\otimes\psi(z), x) = \bigwedge_{z\in A} \hom(\psi(z), a(h(z),x)).$$

In particular, we have that the $\mathcal{V}$-category $\mathcal{V}$ is cocomplete, as well as all of its powers $\mathcal{V}^S$.

A $\mathcal{V}$-functor $f\colon X \to Y$ between cocomplete $\mathcal{V}$-categories is called ***cocontinuous*** whenever f preserves all colimits of weighted colimit diagrams; which means that f is cocontinuous if and only if f preserves copowers and order-theoretic suprema.

Definition 2.2.10. A $\mathcal{V}$-category is ***finitely cocomplete*** if it has all colimits of weighted colimit diagrams whose underlying set of the domain of the weight is finite. We call a $\mathcal{V}$-functor between finitely cocomplete $\mathcal{V}$-categories ***finitely cocontinuous*** if those colimits are preserved.

Therefore:

- A $\mathcal{V}$-category X is finitely cocomplete if and only if X has all copowers, a bottom element, all order-theoretic binary suprema and, moreover, all $\mathcal{V}$-functors $a(-,x)\colon X \to \mathcal{V}^{\mathrm{op}}$ preserve these suprema.

- A map between finitely cocomplete $\mathcal{V}$-categories is a finitely cocontinuous $\mathcal{V}$-functor if and only if it preserves copowers and binary suprema. Note that, by Remark 2.2.7, the preservation of copowers guarantees the preservation of the bottom element.

In the sequel we write $\mathcal{V}$-FinSup to denote the category of separated finitely cocomplete $\mathcal{V}$-categories and finitely cocontinuous $\mathcal{V}$-functors. We also recall that the inclusion functor $\mathcal{V}\text{-}\mathsf{FinSup} \to \mathcal{V}\text{-}\mathsf{Cat}$ is monadic; in particular, this means that $\mathcal{V}$-FinSup is complete and that $\mathcal{V}\text{-}\mathsf{FinSup} \to \mathcal{V}\text{-}\mathsf{Cat}$ preserves limits.

Remark 2.2.11. The interpretation of finitely cocomplete $\mathcal{V}$-categories as certain ordered sets equipped with an action of $\mathcal{V}$ highlights that the category $\mathcal{V}$-FinSup can be seen as a quasivariety (for more information on algebraic categories see [Adámek and Rosický, 1994] and [Adámek et al., 2010]). The first step is to observe that a separated order set with finite suprema is characterised algebraically as a set X equipped with a nullary operation $\bot$ and a binary operation $\vee$ (for example, see [Johnstone, 1986]), subject to the following equations:

$$x \vee x = x, \qquad x \vee y = y \vee x, \qquad x \vee (y \vee z) = (x \vee y) \vee z, \qquad x \vee \bot = x.$$

Then, to describe the action of $\mathcal{V}$, we need to add for every $u \in \mathcal{V}$ a unary operation $- \otimes u$, the equations

$$x \otimes k = x, \quad (x \otimes u) \otimes v = x \otimes (u \otimes v), \quad \bot \otimes u = \bot, \quad (x \vee y) \otimes u = (x \otimes u) \vee (y \otimes u)$$

and, for all $x \in X$ and $S \subseteq \mathcal{V}$ with $v = \bigvee S$ to somehow impose the conditions

$$x \otimes v = \bigvee_{u \in S} (x \otimes u);$$

however, the latter conditions are not formulated using just the operations above. Still, writing $x \leq y$ as an abbreviation for the equation $y = x \vee y$, we can express the condition "$x \otimes v$ is the supremum of $\{x \otimes u \mid u \in S\}$" by the covert equational conditions

$$x \otimes u \leq x \otimes v, \qquad (u \in S)$$

and the implication

$$\bigwedge_{u \in S} (x \otimes u \leq y) \Longrightarrow (x \otimes v \leq y).$$

The last step is to observe that the next equations encode the preservation of finite suprema by the functors $a(-, x) \colon X \to \mathcal{V}^{\mathrm{op}}$

$$\bot \otimes u = \bot, \qquad\qquad (x \vee y) \otimes u = (x \otimes u) \vee (y \otimes u).$$

The morphisms of $\mathcal{V}$-FinSup correspond precisely to the maps preserving these operations.

Therefore, with λ denoting the smallest regular cardinal larger than the size of $\mathcal{V}$, the category $\mathcal{V}$-FinSup is equivalent to a λ-ary quasivariety. From that we conclude that $\mathcal{V}$-FinSup is also cocomplete. Finally, whenever the underlying lattice of the quantale $\mathcal{V}$ is the lattice $[0,1]$, it is enough to consider countable subsets $S \subseteq \mathcal{V}$; therefore, $\mathcal{V}$-FinSup is equivalent to a $\aleph_1$-ary quasivariety.

Another important class of colimit weights is the class of all right adjoint $\mathcal{V}$-distributors with codomain G.

Definition 2.2.12. A $\mathcal{V}$-category X is called ***Cauchy-complete*** if every diagram $(h\colon A \to X, \psi\colon A \looparrowright G)$ with ψ right adjoint has a colimit in X.

The designation "Cauchy-complete" has its roots in Lawvere's amazing observation that, for metric spaces interpreted as $[0,\infty]$-categories, this notion coincides with the classical notion of Cauchy-completeness (see [Lawvere, 1973]). We hasten to remark that every $\mathcal{V}$-functor preserves colimits weighted by right adjoint $\mathcal{V}$-distributors.

In this context, [Hofmann and Tholen, 2010] introduces a closure operator $\overline{(-)}$ on $\mathcal{V}$-Cat which facilitates working with Cauchy-complete $\mathcal{V}$-categories. As usual, a subset $M \subseteq X$ of a $\mathcal{V}$-category (X,a) is ***closed*** whenever $M = \overline{M}$ and is ***dense*** in X whenever $\overline{M} = X$. Below we recall the relevant facts about this closure operator.

Theorem 2.2.13. *The following assertions hold.*

1. *For every $\mathcal{V}$-category (X,a), $x \in X$ and $M \subseteq X$, $x \in \overline{M} \iff k \le \bigvee_{z \in M} a(x,z) \otimes a(z,x)$.*
2. *If $\mathcal{V}$ is completely distributive (see [Raney, 1952] and [Wood, 2004]) with totally below relation $\ll$ and $k \le \bigvee_{u \ll k} u \otimes u$, then $x \in \overline{M}$ if and only if, for every $u \ll k$, there is some $z \in M$ with $u \le a(x,z)$ and $u \le a(z,x)$. By [Flagg, 1992, Theorem 1.12], the quantale $\mathcal{V}$ satisfies $k \le \bigvee_{u \ll k} u \otimes u$ provided that the subset $A = \{u \in \mathcal{V} \mid u \ll k\}$ of $\mathcal{V}$ is directed.*
3. *The $\mathcal{V}$-category $\mathcal{V}$ is Cauchy-complete.*
4. *The full subcategory of $\mathcal{V}$-Cat defined by all Cauchy-complete $\mathcal{V}$-categories is closed under limits in $\mathcal{V}$-Cat.*
5. *Let X be a Cauchy-complete and separated $\mathcal{V}$-category and $M \subseteq X$. Then the $\mathcal{V}$-subcategory M of X is Cauchy-complete if and only if the subset $M \subseteq X$ is closed in X.*

The notion of weighted colimit is dual to the one of weighted limit, of the latter we only need the special case of u-powers, with $u \in \mathcal{V}$.

Definition 2.2.14. A $\mathcal{V}$-category (X,a) is called ***$\mathcal{V}$-powered*** if for every $x \in X$, the $\mathcal{V}$-functor $a(-,x)\colon (X,a)^{\mathrm{op}} \to (\mathcal{V}, \hom)$ has a left adjoint in $\mathcal{V}$-Cat.

Elementwise, this amounts to saying that, for every $x \in X$ and every $u \in \mathcal{V}$, there is an element $x \pitchfork u \in X$, called the ***u-power*** of x, that for every $y \in X$ satisfies

$$\hom(u, a(y, x)) = a(y, x \pitchfork u).$$

The $\mathcal{V}$-category $\mathcal{V}$ is $\mathcal{V}$-powered where $w \pitchfork u = \hom(u, w)$; more generally, $\mathcal{V}^S$ is $\mathcal{V}$-powered with $(h \pitchfork u)(x) = \hom(u, h(x))$, for all $h \in \mathcal{V}^S$, $u \in \mathcal{V}$ and $x \in S$.

Remark 2.2.15. For every $\mathcal{V}$-functor $f : X \to Y$, $x \in X$ and $u \in \mathcal{V}$, $f(u \pitchfork x) \leq u \pitchfork f(x)$.

2.2.1 Continuous quantale structures on the unit interval

In this thesis we are particularly interested in quantales based on the complete lattice $[0, 1]$. Here, we succinctly review the classification of all *continuous* quantale structures $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ on $[0, 1]$ with neutral element 1 and the usual euclidean topology; such quantale structures are also known as continuous t-norms. The results obtained in [Faucett, 1955] and [Mostert and Shields, 1957] show that every such tensor is a combination of the three structures mentioned in Examples 2.2.4 (3). A more detailed presentation of this material is in [Alsina et al., 2006].

We start by recalling some standard notation. An element $x \in [0, 1]$ is called ***idempotent*** whenever $x \otimes x = x$ and ***nilpotent*** whenever $x \neq 0$ and, for some $n \in \mathbb{N}$, $x^n = 0$. The number of idempotent and nilpotent elements characterises the three tensors $\wedge$, $\odot$ and $\otimes$ on $[0, 1]$ among all continuous t-norms.

Proposition 2.2.16. *Assume that* 0 *and* 1 *are the only idempotent elements of* $[0, 1]$ *with respect to a given continuous t-norm. If*

1. $[0, 1]$ *has no nilpotent elements, then* $\otimes = *$ *is multiplication.*
2. $[0, 1]$ *has a nilpotent element, then* $\otimes = \odot$ *is the Łukasiewicz tensor. In this case, every element* x *with* $0 < x < 1$ *is nilpotent.*

To deal with the general case, for a continuous t-norm $\otimes$ consider the subset $E = \{e \in [0, 1] \mid e \text{ is idempotent}\}$. Note that E is closed in $[0, 1]$ since it is an equaliser of the diagram

$$[0,1] \underset{-\otimes-}{\overset{\text{identity}}{\rightrightarrows}} [0,1]$$

in $\mathsf{CompHaus}$.

Lemma 2.2.17. *Let* $\otimes$ *be a continuous t-norm on* $[0, 1]$*,* $x, y \in [0, 1]$ *and* $e \in E$ *such that* $x \leq e \leq y$*. Then* $x \otimes y = x$*.*

Corollary 2.2.18. *Let* $\otimes$ *be a continuous t-norm on* $[0, 1]$ *such that every element is idempotent. Then* $\otimes = \wedge$*.*

Before the main result of this section, note that for idempotents $e < f$ in $[0,1]$, the closed interval $[e,f]$ is a quantale with tensor defined by the restriction of the tensor on $[0,1]$ and neutral element f.

Theorem 2.2.19 ([Mostert and Shields, 1957, Theorem B])**.** *Let $\otimes$ be a continuous t-norm on $[0,1]$. For every non-idempotent $x \in [0,1]$, there exist idempotent elements $e, f \in [0,1]$, with $e < x < f$, such that the quantale $[e,f]$ is isomorphic to the quantale $[0,1]$ with either multiplication or Łukasiewicz tensor.*

Remark 2.2.20. Every isomorphism $[e,f] \to [0,1]$ of quantales is necessarily a homeomorphism.

The following consequence of Theorem 2.2.19 will be particularly useful in Chapter 3.

Corollary 2.2.21. *Let $(u,v) \in [0,1] \times [0,1]$ with $u \otimes v = 0$. Then either $u = 0$ or for some $n \in \mathbb{N}$, $v^n = 0$. Therefore, if there are no nilpotent elements, then $u = 0$ or $v = 0$.*

Proof. Assume $u > 0$. The assertion is clear if there is some idempotent e with $0 < e \leq u$. If there is no $e \in E$ with $0 < e \leq u$, then there is some $f \in E$ with $u < f$ and $[0,f]$ is isomorphic to $[0,1]$ with either multiplication or Łukasiewicz tensor. Since $u \otimes v = 0$, $v < f$. If $[0,f]$ is isomorphic to $[0,1]$ with multiplication, then $v = 0$; otherwise there is some $n \in \mathbb{N}$ with $v^n = 0$. □

In conclusion, the results of this section show that every continuous t-norm on $[0,1]$ is obtained as a combination of infimum, multiplication and Łukasiewicz tensor. Conversely, the next theorem (see [Alsina et al., 2006, Theorem 2.4.2]) states that piecewise combinations of these structures produce continuous t-norm.

Theorem 2.2.22. *Let $\otimes_i$ $(i \in I)$ be a family of continuous quantale structures on $[0,1]$ with neutral element 1 and let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be families of elements of $[0,1]$ such that the open intervals $]a_i, b_i[$ are pairwise disjoint. These data defines a continuous quantale structures on $[0,1]$*

$$x \otimes y = \begin{cases} a_i + (b_i - a_i) \cdot \left(\left(\frac{x-a_i}{b_i-a_i} \right) \otimes_i \left(\frac{y-a_i}{b_i-a_i} \right) \right) & \text{if } x, y \in [a_i, b_i], \\ x \wedge y & \text{otherwise} \end{cases}$$

with neutral element 1 where

$$\hom(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ a_i + (b_i - a_i) \cdot \hom_i \left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i} \right) & \text{if } x, y \in [a_i, b_i], \\ y & \text{otherwise.} \end{cases}$$

2.3 Order and topology

Every topological space comes equipped with a ***natural order*** defined by $x \leq y$ if x belongs to every neighbourhood of y; in other words, if the principal ultrafilter $\dot{x}$ converges to

y. This construction determines the coreflector in the "fundamental adjunction linking order and topology" [Tholen, 2009]

$$\mathsf{Ord} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{Top},$$

where ordered sets are interpreted as topological spaces using the Alexandroff topology. This way, we can study topology with the help of order theoretic notions. But this only makes sense for spaces below the T_1 separation axiom, as T_1-spaces are already characterised by having a discrete natural order. In this work we are mainly interested in stably compact spaces, a class of T_0-spaces "in which compact sets behave in the same way as in the Hausdorff setting" Jung [2004]. This section collects some properties about them with special focus on their alter ego: Nachbin's separated order compact spaces [Nachbin, 1965]. A detailed overview of the concepts and constructions presented here can be found in[Jung, 2004] and [Lawson, 2011]. We begin by introducing some nomenclature.

To distinguish between order closed and topological closed sets we say that a subset A of an ordered set $(X, \leq)$ is ***upper*** if

$$A = \uparrow A = \{x \in X \mid \exists a \in A : a \leq x\}$$

and ***lower*** whenever

$$A = \downarrow A = \{x \in X \mid \exists a \in A : x \leq a\}.$$

A subset A of a topological space X is ***saturated*** if it is a lower set with respect to the natural order of X; in purely topological terms this means that intersecting all open subsets that contain A results in the subset A.

Definition 2.3.1. A ***stably compact*** space is a T_0-space that is:

- ***locally compact*** - every open neighbourhood of a point x contains a compact neighbourhood of x;
- ***coherent*** - the finite intersection of compact saturated sets is compact;
- ***well-filtered*** - if the intersection of a down-directed family of compact saturated sets is contained in an open set U, then some member of the family is already contained in U.

Remark 2.3.2. Being coherent, every stably compact space is compact. Moreover, in the definition of locally compact we can consider without loss of generality that the compact neighbourhood is saturated.

Stably compact spaces are the objects of the category

$$\mathsf{StablyComp},$$

whose morphisms are the continuous maps with the property that the preimage of a compact saturated subset is compact. A map of this kind between stably compact spaces is called a ***spectral*** map; the designation *proper map* is also used in the literature (for instance in [Gierz et al., 2003]) but clashes with the classical notion of proper map in topology (see [Bourbaki, 1966]).

Examples 2.3.3. The notion of stably compact space is rich in familiar examples.

- The category CompHaus of compact Hausdorff spaces and continuous maps is a full subcategory of the category StablyComp.
- A ***spectral space*** is a T_0, well-filtered space where the compact open subsets are closed under finite intersections and form a base for the topology. We denote by Spec the category of spectral spaces and spectral maps.
- A ***Boolean space***, also known as Stone space in the literature, is a Hausdorff spectral space. Boolean spaces and continuous functions form a category that we identify by BooSp.

Theorem 2.3.4. *The category* StablyComp *is complete and wellpowered. The inclusion functor* StablyComp $\to$ Top *preserves limits, finite coproducts and preserves and reflects monocones.*

Proof. It is straightforward to check that the finite coproduct of stably compact spaces is stably compact (see [Goubault-Larrecq, 2013, Proposition 9.2.1]). The other claims follow from monadicity of StablyComp $\to$ Top which is shown in Simmons [1982], where stably compact spaces are studied under the designation of *well-compacted* spaces. □

There is a close connection between stably compact spaces and Nachbin's separated ordered compact spaces [Nachbin, 1950] which was first exposed in [Gierz et al., 1980].

Definition 2.3.5. A ***separated ordered compact space*** consists of a compact space X equipped with a separated order $\leq$ such that the set

$$\{(x,y) \in X \times X \mid x \leq y\}$$

is a closed subset of the product space $X \times X$.

Remark 2.3.6. The function defined by $(x,y) \mapsto (y,x)$ is an automorphism of $X \times X$ in Top, thus the dual order $\geq$ is closed in $X \times X$. This implies that every separated ordered compact space is Hausdorff since the diagonal

$$\Delta = \{(x,y) \mid x \leq y\} \cap \{(x,y) \mid y \leq x\}$$

is a closed subset of $X \times X$.

We denote by

$$\mathsf{SepOrdComp}$$

the category of separated ordered compact spaces and monotone continuous maps. The category SepOrdComp is isomorphic to the category StablyComp, for details see [Gierz et al., 2003]. Under this isomorphism, a separated ordered compact space X corresponds to the stably compact space with the same underlying set and with open sets the open lower subsets of X. In the reverse direction, a stably compact space X defines a separated ordered compact space whose order relation is the natural order of X, and whose compact Hausdorff topology is the so called (generalised) patch topology that is generated by the open subsets and the complements of the compact saturated subsets of X. We will jump freely between both descriptions.

Given a separated ordered compact space X, keeping its topology but taking now its dual order produces another separated ordered compact space, called the ***dual space of*** X and denoted by X^{op}. The separated ordered compact space $[0,1]$ with the Euclidean topology and the usual "less or equal" relation plays a special role in the theory of ordered compact spaces as we will see next. Note that $[0,1]$ is isomorphic to its dual separated ordered compact space $[0,1]^{\mathrm{op}}$. Below we collect some facts about these structures which are in, or follow from, [Nachbin, 1965, Proposition 4 and Theorems 1, 4 and 6].

Proposition 2.3.7. *If A is a compact subset of a separated ordered compact space, then the sets $\uparrow A$ and $\downarrow A$ are closed.*

Corollary 2.3.8. *For every subset A of a separated ordered compact space, the set $\uparrow\overline{A}$ is the smallest closed upper subset that contains A.*

Proposition 2.3.9 (order Urysohn lemma)**.** *Let A and B be subsets of a separated ordered compact space X such that A is a closed upper set, B is a closed lower set and $A \cap B = \varnothing$. Then there is a continuous and monotone function $\psi\colon X \to [0,1]$ such that $\psi(x) = 1$ for every $x \in A$, and $\psi(x) = 0$ for every $x \in B$.*

Corollary 2.3.10. *Every separated ordered compact space satisfies a separation condition on each pair of its points: if $x \nleq y$ then there is an open upper set that contains x and a open lower set that contains y that are disjoint.*

Proposition 2.3.11 (order Tietze extension theorem)**.** *Let A be a closed subset of a separated ordered compact space X. Every continuous and monotone $[0,1]$-valued function on A can be extended to a continuous and monotone $[0,1]$-valued function on X.*

Using the results above, we are able to characterise epimorphisms and regular monomorphisms in SepOrdComp.

Proposition 2.3.12. *The regular monomorphisms in* SepOrdComp *are precisely the embeddings.*

Proof. Clearly, every regular monomorphism is an embedding. We show that the converse implication follows from Proposition 2.3.11. Let $f\colon X \to Y$ be an embedding in $\mathsf{SepOrdComp}$ and assume that $z \notin A$ where $A = f[X]$. Consider the sets

$$A_0 = A \cap {\downarrow}z, \qquad\qquad A_1 = A \cap {\uparrow}z.$$

The sets A_0 and A_1 are closed and every element of A_0 is strictly below every element of A_1. Therefore the map

$$g\colon A_0 \cup A_1 \longrightarrow [0,1],\, x \longmapsto \begin{cases} 0 & \text{if } x \in A_0, \\ 1 & \text{if } x \in A_1 \end{cases}$$

is monotone and continuous. By Proposition 2.3.11, g can be extended to a continuous and monotone map $g\colon A \to [0,1]$, and, with $g_0(z) = 0$ respectively $g_1(z) = 1$, g extends to continuous and monotone maps $g_0, g_1\colon A \cup \{z\} \to [0,1]$. Applying Proposition 2.3.11 again yields morphisms $g_0, g_1\colon Y \to [0,1]$, therefore we can construct morphisms $g_0, g_1\colon Y \to [0,1]$ with $g_0(z) \neq g_1(z)$ that coincide on the elements of A. □

Corollary 2.3.13. *The epimorphisms in* $\mathsf{SepOrdComp}$ *are precisely the surjections.*

Proof. Clearly, every surjective morphism of $\mathsf{SepOrdComp}$ is an epimorphism. Let $f\colon X \to Y$ be an epimorphism in $\mathsf{SepOrdComp}$. Consider its factorisation $f = m \cdot e$ in $\mathsf{SepOrdComp}$ with e surjective and m an embedding. Since m is a regular monomorphism and an epimorphism, we conclude that m is an isomorphism and therefore f is surjective. □

Theorem 2.3.14. *The category* $\mathsf{SepOrdComp}$ *is cocomplete and has an (Epi, RegMono)-factorisation structure.*

Proof. The first claim follows from [Tholen, 2009, Corollary 2], the second from Proposition 2.3.12 and Corollary 2.3.13. □

To conclude this section, we discuss some properties of cones of stably compact spaces. The following definition is quite general. The paradigmatic example comes from the functor $\mathsf{Top} \to \mathsf{Set}$.

Definition 2.3.15. Let $F\colon \mathsf{A} \to \mathsf{B}$ be a functor. A cone $\mathcal{C} = (C \to X_i)_{i\in I}$ in A is said to be ***initial with respect to*** F if for every cone $\mathcal{D} = (D \to X_i)_{i\in I}$ and every morphism $h\colon FD \to FC$ such that $F\mathcal{D} = F\mathcal{C} \cdot h$, there exists a unique A-morphism $\bar{h}\colon D \to C$ with $\mathcal{D} = \mathcal{C} \cdot \bar{h}$ and $h = F\bar{h}$.

We simply say that the cone is ***initial*** whenever no ambiguities arise.

Examples 2.3.16. 1. A cone $(f_i\colon X \to X_i)$ in Top is initial with respect to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$ precisely when X is equipped with the so called initial (weak) topology; that is, the topology of X is generated by the subbasis

$$f_i^{-1}(U) \qquad (i \in I, U \subseteq X_i \text{ open}).$$

2. A monocone of compact Hausdorff spaces and continuous maps is initial in Top (for example, see [Goubault-Larrecq, 2013, Theorem 4.4.27]). The converse also holds, as a initial cone in Top whose domain is a T_0 space is necessarily mono.
3. A monocone in $\mathsf{StablyComp}$ is initial with respect to the forgetful functor $\mathsf{StablyComp} \to \mathsf{Set}$ if and only if is initial with respect to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$.
4. A cone $(f_i \colon (X, \leq) \to (X_i, \leq_i))$ in Ord is initial with respect to the forgetful functor $\mathsf{Ord} \to \mathsf{Set}$ if and only if for every pair of elements $x, y \in X$, the order of X satisfies the condition: $f_i(x) \leq f_i(y)$ for every $i \in I$ implies $x \leq y$.
5. A monocone in $\mathsf{SepOrdComp}$ is initial with respect to the forgetful functor into $\mathsf{CompHaus}$ if and only if it is initial with respect to the functor $\mathsf{Ord} \to \mathsf{Set}$.
6. A monocone in $\mathsf{SepOrdComp}$ is initial with respect to the forgetful functor into Set if and only if is initial with respect to the forgetful functor into $\mathsf{CompHaus}$, by Example 2, if and only if is initial with respect to the functor $\mathsf{Ord} \to \mathsf{Set}$. The previous statements are equivalent to the corresponding cone of stably compact spaces being initial with respect to $\mathsf{Top} \to \mathsf{Set}$.

Remark 2.3.17. In Example 2.3.16(1) the subbasis is actually a basis if the cone is codirected.

Theorem 2.3.18. *Let $F \colon \mathsf{A} \to \mathsf{B}$ be a limit preserving faithful functor and $D \colon \mathsf{I} \to \mathsf{A}$ a diagram. A cone $\mathcal{C}$ for D is a limit of D if and only if the cone $F\mathcal{C}$ is a limit of FD and $\mathcal{C}$ is initial with respect to F.*

Proof. For instance, see [Adámek et al., 1990, Proposition 13.15]. □

Definition 2.3.19. Let A be a category equipped with a faithful functor $U \colon \mathsf{A} \to \mathsf{Set}$. An A-object C is an ***initial cogenerator*** in A if for every A-object A, the cone $(f \colon A \to G)_{f \in \mathsf{A}(A,C)}$ is point-separating and initial with respect to U.

Examples 2.3.20. The categories BooSp, Spec and $\mathsf{CompHaus}$ are the largest full subcategories of $\mathsf{StablyComp}$ where the two-element discrete space, the Sierpiński space and the space $[0, 1]$ equipped with the Euclidean topology are initial-cogenerators in the respective categories.

Regarding the category of all stably compact spaces, we obtain

Proposition 2.3.21. *The separated ordered compact spaces $[0, 1]$ and $[0, 1]^{\mathrm{op}}$ are initial cogenerators in the category $\mathsf{SepOrdComp}$.*

Proof. Immediate consequence of Example 6 and Proposition 2.3.9. □

The description of the categories BooSp, CompHaus and Spec of the previous example makes it easy to identify their image under the isomorphism StablyComp $\simeq$ SepOrdComp. Not surprisingly, compact Hausdorff (Stone) spaces are discrete ordered compact Hausdorff (Stone) spaces. The image of the category Spec is far more interesting: it is the full subcategory of SepOrdComp where the ordered Boolean space $\{0 \leq 1\}$ is an initial cogenerator. In other words, it is the full subcategory of SepOrdComp determined by the spaces where we can formulate the separation condition of Corollary 2.3.10 in terms of clopens instead of opens; we denote this category by Priest as these spaces are typically known as Priestley spaces.

Theorem 2.3.22 ([Priestley, 1970, 1972])**.** *The categories* Spec *and* Priest *are isomorphic.*

Another advantage of the description of Examples 2.3.20 is that it also makes it easy to reason about limits.

Lemma 2.3.23. *The categories* BooSp*,* CompHaus *and* Spec *are closed under initial cones in* StablyComp*; that is, if the codomain of an initial cone in* StablyComp *lives in one of the subcategories considered then also the domain lives in that subcategory. Therefore, the inclusion functors from the categories* BooSp*,* CompHaus *and* Spec *into* StablyComp *create limits.*

Proof. Follows from Examples 2.3.20 since initial cones are closed under composition. □

Furthermore, every limit preserving functor from a complete category that admits a cogenerator is "always" a right adjoint.

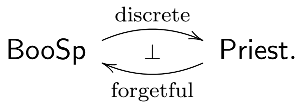
Proposition 2.3.24. *The categories* BooSp*,* CompHaus *and* Spec *are reflective subcategories of the category* StablyComp*. The category* BooSp *is also a reflective subcategory of the category* Spec*.*

Proof. The claim follows from the Special Adjoint Functor Theorem. By Proposition 2.3.23 every subcategory considered is complete and the inclusion functors preserve limits. Moreover, since in each of these cases, the forgetful functor into Set is representable by the one-element space, the injective morphisms are precisely the monomorphisms and, therefore, coincide with monomorphisms in StablyComp. This implies that each subcategory is wellpowered. Finally, in Examples 2.3.20 we saw that the categories have a cogenerator. □

A quick calculation reveals that the inclusion functors CompHaus $\to$ StablyComp and BooSp $\to$ Spec are also left adjoints to the corresponding functors that "take the patch of a space". In the language of ordered compact spaces, this reads as

$$\mathsf{CompHaus} \underset{\text{forgetful}}{\overset{\text{discrete}}{\underset{\bot}{\rightleftarrows}}} \mathsf{SepOrdComp},$$

and

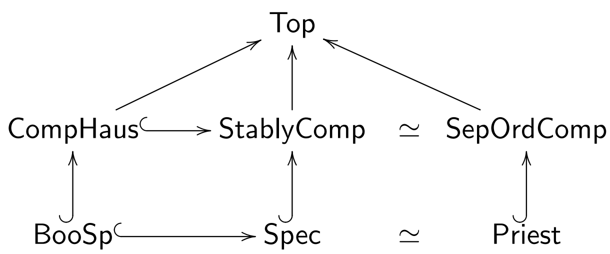


Therefore, by recalling that the category Spec is closed under finite coproducts in the category StablyComp (for instance, see [Goubault-Larrecq, 2013, page 433]), we collect the following useful properties:

Proposition 2.3.25. *The categories* BooSp*,* CompHaus *and* Spec *are complete, cocomplete, wellpowered and inherit the* $(Surjective, Embedding)$*-factorisation structure from the category* StablyComp*. The inclusion functors into* StablyComp *preserve finite coproducts.*

Remark 2.3.26. The regular monomorphisms in Spec are precisely the subspace embeddings. A proof without referring to Priestley duality (see [Priestley, 1970, 1972]) can be found in [Hofmann, 1999]. Therefore, with the same argument of Corollary 2.3.13, we get that the factorisation structures of the proposition above are actually $(Epi, RegMono)$-factorisation structures.

To conclude, we summarise the relationship between the categories introduced in this section in the diagram below.



2.4 Vietoris monads

The Vietoris construction has its roots in [Vietoris, 1922] and various generalisations of this "power construction" are extensively studied in [Schalk, 1993]. At a categorical level, a particular variant has been characterised in concrete categories equipped with a closure operator in [Clementino and Tholen, 1997]. As a source of topological analogues of the powerset monad, the Vietoris construction plays an important role in the results presented in Chapters 3 and 4. In this section we discuss two variants of Vietoris monads and their restrictions to the subcategories of stably compact spaces introduced in the previous section.

For a topological space X, the ***lower Vietoris*** space VX is the *hyperspace* of closed subsets of X equipped with the ***lower Vietoris*** topology that is generated by declaring that for every open set $U \subseteq X$ the set

$$U^{\Diamond} = \{A \in VX \mid A \cap U \neq \varnothing\}$$

is open. This space is seldom studied by itself. Arguably, because topologists are usually interested in establishing properties connecting a space and its hyperspace. Indeed, from this perspective it is not a very interesting space. For instance, independently of the properties of the space X, the space VX is always compact, and it is T_1 precisely when X is the empty space. The lower Vietoris topology is typically introduced together with the ***upper Vietoris*** topology that is generated by requiring that for every open set $U \subseteq X$, the set

$$U^{\Box} = \{A \in VX \mid A \subseteq U\}$$

is open; instead of studying the lower or the upper Vietoris topologies individually, usually, topologists are far more interested in the ***Vietoris topology*** that is the supremum of both them, and was first introduced by Vietoris[Vietoris, 1922] in the context of compact Hausdorff spaces. Nevertheless, as the example below shows, the lower Vietoris space is the only construction mentioned above that becomes an endofunctor on Top by mapping a continuous function $X \to Y$ to the function $Vf\colon VX \to VY$, defined by $A \mapsto \overline{fA}$; we call it the ***lower Vietoris functor***. In fact, this functor is even part of a monad $\mathbb{V} = (\mathsf{V}, e, m)$ on Top (see [Schalk, 1993, Section 6.3]), the so called ***lower Vietoris monad***. The X-component of the unit $e\colon X \to VX$ is defined by $x \mapsto \uparrow\overline{\{x\}}$ and the X-component of the multiplication $m\colon \mathsf{VV}X \to VX$ is given by $\mathcal{A} \mapsto \bigcup \mathcal{A}$.

Remark 2.4.1. The Vietoris topology on the hyperspace of closed subsets does not define an obvious functor on Top. Consider the set $\{1,2,3\}$ equipped with the topology generated by the sets $\{1,2\}$ and $\{2,3\}$. For the subspace embedding $i\colon \{1,2\} \to \{1,2,3\}$, $(Vi)^{-1}[\{1,2\}^{\Box}] = \{\varnothing, \{1\}\}$. However, every open set of $\mathsf{V}\{1,2\}$ that contains $\{1\}$ contains $\{1,2\}$.

The remark above shows that it is naive to move directly the definition of Vietoris space from $\mathsf{CompHaus}$ to Top. Alternatively, we can use that closed subsets of compact Hausdorff spaces correspond exactly to compact subsets. The ***compact Vietoris functor*** sends a space X to the hyperspace VX of compact subsets of X with topology generated by declaring that for every open set $U \subseteq X$ the sets

$$U^{\Diamond} = \{A \in VX \mid A \cap U \neq \varnothing\}$$
$$U^{\Box} = \{A \in VX \mid A \subseteq U\}$$

are open. As the lower Vietoris functor, the compact Vietoris functor is also part of a monad, but with unit and multiplication defined as in the powerset monad; the monads are seemingly unrelated on Top, yet, as we will see in the remainder of the section, they are closely related when restricted to $\mathsf{StablyComp}$ and $\mathsf{CompHaus}$ respectively.

Proposition 2.4.2 ([Schalk, 1993])**.** *The lower Vietoris monad restricts to the category* $\mathsf{StablyComp}$ *and* Spec.

Since the category $\mathsf{StablyComp}$ is isomorphic to the category $\mathsf{SepOrdComp}$ of separated ordered compact spaces and monotone maps (see Section 2.3) we obtain a new Vietoris monad on the category $\mathsf{SepOrdComp}$ that we denote by $\mathbb{V} = (V\colon \mathsf{SepOrdComp} \to \mathsf{SepOrdComp}, m, e)$.

Proposition 2.4.3. *Under the isomorphism* $\mathsf{StablyComp} \simeq \mathsf{SepOrdComp}$*, the lower Vietoris monad on* $\mathsf{StablyComp}$ *corresponds to the monad*

$$\mathbb{V} = (V\colon \mathsf{SepOrdComp} \to \mathsf{SepOrdComp}, m, e)$$

whose functor

$$\mathsf{V}\colon \mathsf{SepOrdComp} \to \mathsf{SepOrdComp}$$

sends a separated ordered compact space X to the space VX of all upper-closed subsets of X, with order containment $\supseteq$, and compact topology generated by the sets

$$\{A \in VX \mid A \cap U \neq \varnothing\} \quad (U \subseteq X \text{ lower-open}), \tag{2.4.i}$$
$$\{A \in VX \mid A \cap K = \varnothing\} \quad (K \subseteq X \text{ lower-closed}).$$

Given a map $f\colon X \to Y$ in $\mathsf{SepOrdComp}$*, the functor returns the map Vf that sends a upper-closed subset $A \subseteq X$ to the up-closure $\uparrow f[A]$ of $f[A]$. The X-component of the unit sends a point of x to the set $\Uparrow\{x\}$ and the X-component of the multiplication maps a subset $\mathcal{A} \subseteq VVX$ to the subset $\bigcup \mathcal{A} \subseteq VX$.*

Proof. Let $(X, \leq, \tau)$ be a separated ordered compact space with corresponding stably compact space (X, σ). Clearly, the underlying set of $\mathsf{V}(X, \sigma)$ is the set of all upper-closed subsets of X. We will show that the patch topology of $\mathsf{V}(X, \sigma)$ coincides with the topology defined by (2.4.i). First note that every set of the form

$$\{A \subseteq X \mid A \text{ upper-closed and } A \cap U \neq \varnothing\} \quad (U \subseteq X \text{ lower-open}),$$

is open in $\mathsf{V}(X, \sigma)$ and, therefore, is also open in the patch topology. For $K \subseteq X$ lower-closed, the complement of the set

$$\{A \subseteq X \mid A \text{ upper-closed and } A \cap K = \varnothing\}$$

is equal to $K^\diamond$. Using Alexander's Subbase Theorem, it is straightforward to verify that $K^\diamond$ is compact in $\mathsf{V}(X, \sigma)$. Since the natural order of $\mathsf{V}(X, \sigma)$ is subset containment, $K^\diamond$ is also saturated. Hence, the topology defined by (2.4.i) is coarser than the patch topology of $\mathsf{V}(X, \sigma)$. Since it is also Hausdorff, by [Jung, 2004, Lemma 2.2], both topologies coincide (see Engelking [1989]). In particular, the construction of the proposition defines indeed a separated ordered compact space.

In regard to maps in $\mathsf{SepOrdComp}$, Proposition 2.3.7 tells that for every map $f\colon X \to Y$

in $\mathsf{SepOrdComp}$ and every upper-closed subset $A \subseteq X$, the up-closure $\uparrow f[A]$ of $f[A]$ is closed in Y, and therefore coincides with the closure of $f[A]$ in the stably compact topology of Y. The description of the unit and multiplication follow by routine calculation. □

Therefore, by transporting the Vietoris monad in $\mathsf{SepOrdComp}$ along the adjunction

$$\mathsf{SepOrdComp} \underset{\text{discrete}}{\overset{\text{forgetful}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{CompHaus},$$

we recover the classical (compact) Vietoris monad on the category of compact Hausdorff spaces; we can even restrict this monad further to the category of Boolean spaces since the Vietoris monad in $\mathsf{SepOrdComp}$ also restricts to Priest, as we have seen in Proposition 2.4.2, and the adjunction above restricts to

$$\mathsf{Priest} \underset{\text{discrete}}{\overset{\text{forgetful}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{BooSp}.$$

Remark 2.4.4. In Chapter 3 we will be interested in the Kleisli categories $\mathsf{SepOrdComp}_{\mathbb{V}}$ and $\mathsf{CompHaus}_{\mathbb{V}}$. A morphism $X \to VY$ in $\mathsf{CompHaus}$ corresponds to a relation $X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, and a morphism $X \to VY$ in $\mathsf{SepOrdComp}$ corresponds to a distributor between the underlying separated ordered sets. In both cases composition in the respective Kleisli categories corresponds to relational composition.

2.5 Coalgebras

Coalgebras [Rutten, 2000; Adámek, 2005], which are duals of algebras, form a powerful theory especialy suited to model transition systems such as Kripke frames, stream automata, or labelled transition systems. In this section we sketch the strategy employed in Section 4.2 to study limits in categories of coalgebras. We start with some categorical notions that the reader may not frequently meet.

Definition 2.5.1. A diagram $D\colon \mathsf{I} \to \mathsf{C}$ is said to be ***codirected*** whenever I is a codirected separated ordered set, that is, I is non-empty and for all $i, j \in \mathsf{I}$ there is some $k \in \mathsf{I}$ with $k \to i$ and $k \to j$. A cone for a codirected diagram is called a ***codirected cone***, and a limit of such diagram is said to be a ***codirected limit***.

Example 2.5.2. 1. Inverse sequence diagrams, which have the shape depicted below, are codirected.

$$\cdot \longleftarrow \cdot \longleftarrow \cdot \longleftarrow \dots$$

Inverse sequence diagrams play a central role in showing that a given functor admits a terminal coalgebra (see Theorem 2.5.11).

2. A codirected limit of a diagram $D\colon \mathsf{I} \to \mathsf{Set}$ is given by the subset

$$\left\{(x_i)_{i\in\mathsf{I}} \in \prod_{i\in\mathsf{I}} D(i) \mid \forall j \to i \in \mathsf{I}, D(j\to i)(x_j) = x_i\right\}$$

of the product $\prod_{i\in\mathsf{I}} D(i)$ together with the restrictions of the projection maps.

Codirectedness plays particularly well with compactness as illustrated in the characterisation of codirected limits of compact Hausdorff spaces below. This result is hinted in [Bourbaki, 1942] and proved in Hofmann [1999] (in German) but it seems to be rarely used in the literature.

Theorem 2.5.3. *Let $D\colon \mathsf{I} \to \mathsf{CompHaus}$ be a codirected diagram and $\mathcal{C} = (p_i\colon L \to D(i))_{i\in\mathsf{I}}$ a cone for D. The following conditions are equivalent:*

1. *The cone $\mathcal{C}$ is a limit of D.*
2. *The cone $\mathcal{C}$ is mono and, for every $i \in \mathsf{I}$, the image of p_i contains the intersection of the images of all $D(j \to i)$, in symbols*

$$\operatorname{im} p_i \supseteq \bigcap_{j\to i} \operatorname{im} D(j \to i).$$

Proof. Assume first that $(p_i\colon L \to D(i))_{i\in\mathsf{I}}$ satisfies the two conditions and let $(f_i\colon X \to D(i))_{i\in\mathsf{I}}$ be a cone for D. Let $x \in X$, and, for every $i \in \mathsf{I}$, put $A_i = p_i^{-1}(f_i(x))$. Clearly, A_i is closed, moreover, A_i is non-empty since

$$\operatorname{im} f_i \subseteq \bigcap_{j\to i} \operatorname{im} D(j \to i) = \operatorname{im} p_i$$

Since the family $(A_i)_{i\in\mathsf{I}}$ is codirected and L is compact, there is some $z \in \bigcap_{i\in\mathsf{I}} A_i$. We put $f(x) = z$, this way we define a map $f\colon X \to L$ with $p_i \cdot f = f_i$, for all $i \in \mathsf{I}$. Since $(p_i\colon L \to D(i))_{i\in\mathsf{I}}$ is a monocone, we conclude that $(p_i\colon L \to D(i))_{i\in\mathsf{I}}$ is a limit of D. Conversely, if $(p_i\colon L \to D(i))_{i\in\mathsf{I}}$ is a limit, then it is clearly a monocone. Let now $i_0 \in \mathsf{I}$ and $x \in \bigcap_{j\to i_0} \operatorname{im} D(j \to i_0)$. We may assume that i_0 is final in I. For each $i \in \mathsf{I}$, we put

$$A_i = \{(x_i)_{i\in\mathsf{I}} \in \prod_{i\in\mathsf{I}} D(i) \mid x_{i_0} = x \text{ and, for all } i \to j \in \mathsf{I}, x_j = D(i \to j)(x_i)\}.$$

Then A_i is non-empty, and it is a closed subset of $\prod_{i\in\mathsf{I}} D(i)$ since it is an equaliser of continuous maps between Hausdorff spaces. Furthermore, for $i \to j \in \mathsf{I}$, $A_i \subseteq A_j$. Hence there is some $z \in \bigcap_{i\in\mathsf{I}} A_i$; by construction, $z \in L$ and $p_{i_0}(z) = x$. □

Remark 2.5.4. For every cone $(p_i\colon C \to D(i))_{i\in \mathsf{I}}$ the inequality $\operatorname{im} p_i \subseteq \bigcap_{j\to i} \operatorname{im} D(j \to i)$ holds. Thus, in the theorem above, the reverse inequality distinguishes monocones from limit cones.

Definition 2.5.5. A category C is said to be ***connected*** if it is non-empty and every two objects $A, B \in \mathsf{C}$ are connected by a finite zig-zag of morphisms as depicted below.

$$A \leftarrow \cdot \rightarrow \cdots \leftarrow \cdot \rightarrow B$$

A diagram $D\colon \mathsf{I} \to \mathsf{C}$ with I connected is called a ***connected diagram*** and a limit of such diagram is said to be a ***connected limit***.

Examples 2.5.6. Equalisers and codirected limits are two examples of connected limits.

As we will see in the remainder of the section, the concepts introduced before play an important role in the theory of coalgebras.

Definition 2.5.7. Let $F\colon \mathsf{C} \to \mathsf{C}$ be a functor. A coalgebra of F, or an F-coalgebra, consists of a C-object X together with a C-morphism of type $X \to FX$.

The collection of coalgebras of a functor form a category in a natural way.

Definition 2.5.8. Let $F\colon \mathsf{C} \to \mathsf{C}$ be a functor. The ***category*** $\mathsf{CoAlg}(F)$ has as objects the F-coalgebras; a morphism $f\colon (A,a) \to (B,b)$ in $\mathsf{CoAlg}(F)$ is a C-morphism such that $Ff \cdot a = b \cdot f$.

We can easily construct colimits in categories of coalgebras from colimits in the base category, for the exact same reason that is easy to construct limits in categories of algebras (for instance, see [Barr and Wells, 1985]).

Theorem 2.5.9. *Let $F\colon \mathsf{C} \to \mathsf{C}$ be a functor. The forgetful functor $\mathsf{CoAlg}(F) \to \mathsf{C}$ creates colimits.*

The study of limits is usually much more complex, except, if the underlying functor preserves those limits.

Theorem 2.5.10. *If a functor $F\colon \mathsf{C} \to \mathsf{C}$ preserves limits of a certain type then the forgetful functor $\mathsf{CoAlg}(F) \to \mathsf{C}$ creates limits of the same type.*

In practise, it is often too much to ask for the functor to preserve all the limits that we are interested in. In particular, in this context it is rare for a functor to preserve terminal objects , nevertheless, with the next result we can determine terminal coalgebras if the functor preserves a specific codirected limit.

Theorem 2.5.11. *Let* C *be a category with a terminal object* 1 *and* $F\colon \mathsf{C} \to \mathsf{C}$ *a functor. If the category* C *has a limit* L *of the diagram*

$$1 \longleftarrow F1 \longleftarrow FF1 \longleftarrow \dots$$

and F *preserves it, then the canonical isomorphism* $L \to FL$ *is a terminal* F*-coalgebra.*

Proof. For instance, see [Adámek, 2005]. □

The next notions and results summarise the strategy employed in Section 4.2 to prove completeness in categories of coalgebras.

Definition 2.5.12. A functor $F\colon \mathsf{C} \to \mathsf{C}$ is said to be a ***covarietor*** if the canonical forgetful functor $\mathsf{CoAlg}(F) \to \mathsf{C}$ is left adjoint.

Theorem 2.5.13. *Let* C *be a cocomplete category with finite limits and limits of countable chains. Every endofunctor on* C *that preserves limits of countable chains is a covarietor.*

Proof. For example, see [Barr and Wells, 1985, Proposition 7 of Section 9.4] □

This adjoint situation allows to take advantage of the theory of (co)monads to simplify proving completeness.

Theorem 2.5.14 ([Linton, 1969])**.** *Let* F *be a covarietor over a complete category. If the category* $\mathsf{CoAlg}(F)$ *has equalisers then it is complete.*

Related to this, Hughes proved the following theorem

Theorem 2.5.15 ([Hughes, 2001, Theorem 2.4.2])**.** *Let* C *be a regularly wellpowered, cocomplete category with equalisers. Moreover, assume that it has an (Epi, RegMono)-factorisation structure, and that the functor* $F\colon \mathsf{C} \to \mathsf{C}$ *preserves regular monomorphisms. Then* $\mathsf{CoAlg}(F)$ *has equalisers.*

Motivated by the previous result, in the sequel we briefly study limits in categories that admit a factorisation structure for cones or morphisms. As a general reference for factorisation structures, see [Adámek et al., 1990].

Definition 2.5.16. Fix a small category I. Consider a category C and a class $\mathcal{M}$ of cones of shape I in C. The category C is $\mathcal{M}$-wellpowered if for every diagram $D\colon \mathsf{I} \to \mathsf{C}$ the collection, up to isomorphism, of cones for D in $\mathcal{M}$ is a set.

The next lemma follows from standard arguments; it is in the spirit of [Adámek et al., 1990, Section 12] and shows that "cocompleteness almost implies completeness".

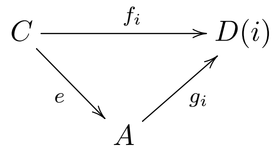
Lemma 2.5.17. *Let* C *be a cocomplete category and* I *a small category. Furthermore, let* E *be a class of* C*-morphisms and* $\mathcal{M}$ *be a class of cones of shape* I *in* C*. If* C *is* $\mathcal{M}$*-wellpowered and every cone of shape* I *has a* $(E, \mathcal{M})$*-factorisation, then* C *has limits of shape* I*.*

Proof. We will show that the diagonal functor

$$\Delta\colon \mathsf{C} \to \mathsf{C}^{\mathsf{I}}$$

has a right adjoint, using Freyd's General Adjoint Functor Theorem (see MacLane [1971]). By assumption C is cocomplete and it is clear that the functor Δ preserves colimits, so we just need to show that the Solution Set Condition holds. In this case it unfolds into the following condition: for every functor $D\colon \mathsf{I} \to \mathsf{C}$, there is a set $\mathcal{S}$ of cones for D such that every cone for D factors through a cone in $\mathcal{S}$.

Since C is $\mathcal{M}$-wellpowered we have a set $\mathcal{S}$ of representants of cones for D in $\mathcal{M}$. This set has the desired property because, by assumption, every cone for I has an $(E, \mathcal{M})$-factorisation, which means that a cone $(f_i\colon C \to D(i))_{i\in\mathsf{I}}$ factors through a cone $(g_i\colon A \to D(i))_{i\in\mathsf{I}}$ in $\mathcal{S}$ as depicted below.



□

In practise it might be easier to obtain factorisations for cones from factorisations for morphisms. The following proposition describes two situations where this is possible and can be applied to many "everyday categories" like Set, Top or even $\mathsf{StablyComp}$.

Proposition 2.5.18. *Fix a category* I. *Consider a category* C *and classes* E *and* M *of* C-*morphisms such that every morphism of* C *is* (E, M)-*factorisable,* E *is contained in the class of* C-*epimorphisms and* C *is* M-*wellpowered.*

If one of the conditions below is satisfied, then there is a class $\mathcal{M}$ *of cones of shape* I *such that every cone of shape* I *is* $(E, \mathcal{M})$-*factorisable and* C *is* $\mathcal{M}$-*wellpowered.*

1. C *has products;*
2. I *is the category* $1 \rightrightarrows 2$.

Proof. In the first case we can choose

$$\mathcal{M} = \left\{ \text{all cones } (f_i\colon X \to D(i))_{i\in\mathsf{I}} \text{ of shape } \mathsf{I} \text{ where } \langle f_i\rangle_{i\in\mathsf{I}}\colon X \to \prod_{i\in\mathsf{I}} D(i) \text{ is in } M \right\}.$$

Then, it is clear that every cone for I is $(E, \mathcal{M})$-factorisable (see [Adámek et al., 1990, Proposition 15.19]), and C is $\mathcal{M}$-wellpowered.

For the second case, it is straightforward to verify that the class of cones

$$\mathcal{M} = \{\text{all cones } (f_i\colon X \to D(i))_{i\in\mathsf{I}} \text{ of shape } \mathsf{I} \text{ with } f_1 \text{ in } M\},$$

makes every cone of shape I $(E, \mathcal{M})$-factorisable and that the category C is $\mathcal{M}$-wellpowered. □

Remark 2.5.19. In the proposition above, if we consider that C is (E, M)-structured we obtain that C is $(E, \mathcal{M})$-structured for cones of shape I.

Corollary 2.5.20. *Let C be a cocomplete category and E and M classes of C-morphisms such that every morphism of C is (E, M)-factorisable, E is contained in the class of C-epimorphisms and C is M-wellpowered. Then, the category C has equalisers.*

Remark 2.5.21. Corollary 2.5.20 above shows that it is redundant to assume the existence of equalisers in Hughes' theorem (Theorem 2.5.15).

Now, to apply Lemma 2.5.17 to categories of coalgebras we can use well-known results that lift factorisation structures from a base category to its categories of coalgebras.

Theorem 2.5.22. *Let I be a small category and F an endofunctor over a cocomplete category C. If C is $(E, \mathcal{M})$-structured for cones of shape I, $\mathcal{M}$-wellpowered and F sends cones in $\mathcal{M}$ to cones in $\mathcal{M}$, then $\mathsf{CoAlg}(F)$ has limits of shape I.*

Proof. The assumptions guarantee that the factorisation system in C lifts to $\mathsf{CoAlg}(F)$ (for instance, see Adámek [2005]; Chen [2014]). The claim then follows from Lemma 2.5.17. □

Combining Proposition 2.5.18, Remark 2.5.19 and Theorem 2.5.22, we obtain

Theorem 2.5.23. *Let F be an endofunctor over a cocomplete category C with products and an (E, M)-factorisation structure such that E is contained in the class of C-epimorphisms and C is M-wellpowered. If F preserves products and sends morphisms in M to morphisms in M, then $\mathsf{CoAlg}(F)$ is complete.*

Theorem 2.5.24. *Let F be an endofunctor over a cocomplete category C that has an (E, M)-factorisation structure such that E is contained in the class of C-epimorphisms and C is M-wellpowered. If F sends morphisms in M to morphisms in M, then $\mathsf{CoAlg}(F)$ has equalisers.*

This result slightly generalises Hughes' theorem and shows that we do not need to assume the existence of equalisers in the base category, although, they always exist as stated in Remark 2.5.21.

Corollary 2.5.25. *Let F be an endofunctor over a cocomplete category C. If C is regularly wellpowered, has an (Epi, RegMono)-factorisation structure and F preserves regular monomorphisms, then $\mathsf{CoAlg}(F)$ has equalisers.*

Finally, the next result summarises the strategy that we will use in Section 4.2 to prove completeness in categories of coalgebras.

Theorem 2.5.26. *Let C be a category that*

- *is complete,*
- *cocomplete,*
- *has an (E, M)-factorisation structure such that* C *is M-wellpowered and E is contained in the class of* C*-epimorphisms.*

If a functor $F\colon \mathsf{C} \to \mathsf{C}$ sends morphisms in M to morphisms in M and preserves codirected limits then the category of coalgebras of F is complete.

Proof. The category C satisfies all the conditions necessary to apply Theorem 2.5.13 and Theorem 2.5.24. Thus, since F preserves codirected limits, it is a covarietor and because preserves morphisms in M the category $\mathsf{CoAlg}(F)$ has equalisers. Therefore, the claim follows by Theorem 2.5.14. □

Chapter 3

Duality theory

The main goal of this chapter is to extend Halmos' dual equivalence to categories including all compact Hausdorff spaces in a way that the objects of the corresponding dual category appear as generalisations of Boolean algebras. Part of this work has already been published in [Hofmann and Nora, 2015] and [Hofmann and Nora, 2018].

Theorem (Halmos' dual equivalence)**.** *The Kleisli category* $\mathsf{BooSp}_{\mathbb{V}}$ *of the Vietoris monad on* BooSp *(see section 2.4) is dually equivalent to the category* $\mathsf{FinSup}_{\mathrm{BA}}$ *of Boolean algebras and finite suprema preserving maps.*

Halmos gives a direct proof for this result in [Halmos, 1956]. He does not, however, talk about Kleisli categories or even about monads; instead, he refers to Boolean relations which happen to correspond precisely to morphisms in $\mathsf{BooSp}_{\mathbb{V}}$ as shown in Kupke et al. [2004]. This observation allows to tackle Halmos duality indirectly with the help of monad theory. We discuss how in Section 3.1, where we approach the problem of deducing in a uniform way duality theorems involving categories of *relations*. In the case of Halmos' duality, our approach highlights the role of the two-element discrete space as an initial cogenerator in the category BooSp. To pass to the category $\mathsf{CompHaus}$ for example, we would need to replace the two-element discrete space with an initial cogenerator of $\mathsf{CompHaus}$ such as the unit interval. Together with the Vietoris functor on $\mathsf{CompHaus}$, this idea by itself could lead us to a Halmos version of Gelfand's duality theorem (see [Gelfand, 1941]).

Theorem (Gelfand's dual equivalence)**.** *The category* $\mathsf{CompHaus}$ *is dually equivalent to the category* C^*-Alg *of* C^**-algebras and homomorphisms.*

But to pass from functions to continuous *relations*, what part of the structure of a C^*-algebra the morphisms need to ignore? The answer is not obvious, and even if it were, at best we would end up with a duality result where the objects of the dual category of $\mathsf{CompHaus}$ do not seem generalisations of Boolean algebras. To improve upon this, we resort to quantale-enriched category theory. Our thesis is that *the passage from the two-element space to the* *compact Hausdorff space* $[0,1]$ *should be matched on the algebraic side of Halmos' duality by a move from ordered structures (*2*-categories) to metric structures (*$[0,1]$*-categories).* In Section 3.2 we explore this idea to develop duality theory for the categories $\mathsf{SepOrdComp}_{\mathbb{V}}$, $\mathsf{SepOrdComp}$ and $\mathsf{CompHaus}$ according to the possible choice of quantale structure on the unit interval. The duality results of this section will be used in Section 4.1 to show that the category of coalgebras of the Vietoris monad on $\mathsf{SepOrdComp}$ is an $\aleph_1$-ary quasivariety.

3.1 The point of view of triples

We start with some well-known results about the structure and construction of dual adjunctions. There is a vast literature on this subject, notably [Lambek and Rattray, 1978, 1979], [Dimov and Tholen, 1989], [Porst and Tholen, 1991], [Johnstone, 1986] and [Clark and Davey, 1998].

Consider an adjunction

(3.1.i) $$\mathsf{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{A}^{\mathrm{op}} \quad (F \dashv G)$$

between a category X and the dual of a category A. In general, such an adjunction is not an equivalence. Nevertheless, as mentioned in Section 2.1.5 one can always consider its restriction to the full subcategories $\mathrm{Fix}(\mathsf{X})$ and $\mathrm{Fix}(\mathsf{A})$ of X and A, defined by the classes of objects

$$\{X \mid \eta_X \text{ is an isomorphism}\} \qquad \text{and} \qquad \{A \mid \varepsilon_A \text{ is an isomorphism}\},$$

where it yields an equivalence $\mathrm{Fix}(\mathsf{X}) \simeq \mathrm{Fix}(\mathsf{A})^{\mathrm{op}}$ (see [Porst and Tholen, 1991]). The passage from X to $\mathrm{Fix}(\mathsf{X})$ is useful only if we keep all the "interesting objects". However, this is not always the case as $\mathrm{Fix}(\mathsf{X})$ can be even empty. *En passant* we mention that these fixed subcategories are reflective in A, respectively in X, if the monad induced by the adjunction (3.1.i) on A, respectively X, is idempotent (see [Lambek and Rattray, 1979, Theorem 2.0] for details).

We can prove the classical Stone duality theorem for Boolean algebras in this way. The categories $\mathsf{CompHaus}$ and BA are linked by the adjunction where the left adjoint sends a compact Hausdorff space to its Boolean algebra of clopen sets, and maps a continuous function f to the algebra homomorphism that "takes the inverse image by f".

Considering the two-element discrete space and the two-element Boolean algebra this amounts to saying that the respective liftings of the $\hom(-,2)$ functors are adjoint.

(3.1.ii)
$$\mathsf{CompHaus} \underset{\hom(-,2)}{\overset{\hom(-,2)}{\rightleftarrows}} \mathsf{BA}^{\mathrm{op}} \quad (\perp)$$

Stone's representation theorem affirms that Fix(A) is actually the category BA and, as we will see later in this section, Fix(X) is the category BooSp of Boolean spaces and continuous maps.

Like other "everyday categories", the categories CompHaus and BA come equipped with faithful representable functors into Set. This property allows to follow categorical guidance further when it comes to construct dual adjunctions. For this reason, throughout this chapter we assume that X and A are equipped with faithful functors

$$|-|\colon \mathsf{X} \longrightarrow \mathsf{Set} \qquad \text{and} \qquad |-|\colon \mathsf{A} \longrightarrow \mathsf{Set}.$$

Definition 3.1.1. The adjunction (3.1.i) is ***induced by the dualising object*** $(\widetilde{X}, \widetilde{A})$, with objects $\widetilde{X}$ in X and $\widetilde{A}$ in A, when $|\widetilde{X}| = |\widetilde{A}|$, $|F| = \hom(-, \widetilde{X})$, $|G| = \hom(-, \widetilde{A})$ and the units are given by

(3.1.iii)
$$\begin{aligned} \eta_X\colon X &\longrightarrow GFX \\ x &\longmapsto \mathrm{ev}_x \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varepsilon_A\colon A &\longrightarrow FGA; \\ a &\longmapsto \mathrm{ev}_a \end{aligned}$$

with ev_x and ev_a denoting the evaluation maps.

If the forgetful functors to Set are representable by objects X_0 in X and A_0 in A, then every adjunction (3.1.i) is of this form, up to natural equivalence (see [Dimov and Tholen, 1989] and [Porst and Tholen, 1991]).

Remark 3.1.2. Consider an adjunction (3.1.i) induced by a dualising object $(\widetilde{X}, \widetilde{A})$. For every $\psi\colon X \to \widetilde{X}$ and $\varphi\colon A \to \widetilde{A}$, the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GFX \\ & \underset{\psi}{\searrow} & \downarrow \mathrm{ev}_\psi \\ & & \widetilde{X} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\varepsilon_A} & FGA \\ & \underset{\varphi}{\searrow} & \downarrow \mathrm{ev}_\varphi \\ & & \widetilde{A} \end{array}$$

commute.

We turn now to the question "How to construct dual equivalences?". Motivated by the considerations above, we assume that $\widetilde{X}$ and $\widetilde{A}$ are objects in X and A respectively, with the same underlying set $|\widetilde{X}| = |\widetilde{A}|$. To obtain a dual adjunction, we need to lift the hom-functors $\hom(-, \widetilde{X})\colon \mathsf{X}^{\mathrm{op}} \to \mathsf{Set}$ and $\hom(-, \widetilde{A})\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Set}$ to functors $F\colon \mathsf{X}^{\mathrm{op}} \to \mathsf{A}$ and $G\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{X}$

in such a way that the maps (3.1.iii) underlie an X-morphism respectively and A-morphism. To this end, we consider the following two conditions.

(Init X) For every object X in X, the cone $(\mathrm{ev}_x\colon \hom(X,\widetilde{X}) \to |\widetilde{A}|, \psi \mapsto \psi(x))_{x\in|X|}$ admits an initial lift $(\mathrm{ev}_x\colon F(X) \to \widetilde{A})_{x\in|X|}$.

(Init A) For every object A in A, the cone $(\mathrm{ev}_a\colon \hom(A,\widetilde{A}) \to |\widetilde{X}|, \psi \mapsto \psi(a))_{a\in|A|}$ admits an initial lift $(\mathrm{ev}_a\colon G(A) \to \widetilde{X})_{a\in|A|}$.

The following result can be found in [Porst and Tholen, 1991].

Theorem 3.1.3. *If conditions (Init X) and (Init A) are fulfilled, then the initial lifts above define the object part of a dual adjunction* (3.1.i) *induced by* $(\widetilde{X},\widetilde{A})$.

Clearly, if the forgetful functors to Set are topological (see [Adámek et al., 1990]), then (Init X) and (Init A) are fulfilled. The following proposition describes a typical situation and it is our main weapon to construct dual adjunctions.

Proposition 3.1.4. *Let* A *be the category of algebras for a signature* Ω *of operation symbols and assume that* X *is complete and* $|-|\colon \mathsf{X} \to \mathsf{Set}$ *preserves limits. Furthermore, assume that, for every operation symbol* $\omega \in \Omega$*, the corresponding operation* $|\widetilde{A}|^I \to |\widetilde{A}|$ *underlies an* X*-morphism* $\widetilde{X}^I \to \widetilde{X}$*. Then both (Init X) and (Init A) are fulfilled.*

Proof. This result is essentially proven in [Lambek and Rattray, 1979, Proposition 2.4]. Firstly, since all operations on $\widetilde{A}$ are X-morphisms, the algebra structure on $\hom(X,\widetilde{X})$ can be defined pointwise. Secondly, for each algebra A, the canonical inclusion $\hom(A,\widetilde{A}) \to |\widetilde{X}|^{|A|}$ is the equaliser of a pair of X-morphisms between powers of $\widetilde{X}$. In fact, a map $f\colon |A| \to |\widetilde{A}|$ is an algebra homomorphism whenever, for every operation symbol $\omega \in \Omega$ with arity I and every $h \in |A|^I$,

$$f(\omega_A(h)) = \omega_{\widetilde{A}}(f\cdot h).$$

In other words, the set of maps $f\colon |A| \to |\widetilde{A}|$ which preserve the operation ω is precisely the equaliser of

$$\pi_{\omega_A(h)}\colon |\widetilde{A}|^{|A|} \longrightarrow |\widetilde{A}|$$

and the composite

$$|\widetilde{A}|^{|A|} \xrightarrow{-\cdot h} |\widetilde{A}|^I \xrightarrow{\omega_{\widetilde{A}}} |\widetilde{A}|.$$

Since both maps underlie X-morphisms $\widetilde{X}^{|A|} \to \widetilde{X}$, the assertion follows. □

Remark 3.1.5. The result above remains valid if

- the objects of A admit an order relation and some of the operations are only required to be preserved laxly, and

- the order relation $R \to |\widetilde{A}| \times |\widetilde{A}|$ of $\widetilde{A}$ underlies an X-morphism $R' \to \widetilde{X} \times \widetilde{X}$.

In fact, using the notation of the proof above, the set of maps $f\colon |A| \to |\widetilde{A}|$ with

$$f(\omega_A(h)) \leq \omega_{\widetilde{A}}(f \cdot h)$$

for all $h \in |A|^I$ can be described as the pullback of the diagram

$$\begin{array}{ccc} & & R \\ & & \downarrow \\ |\widetilde{A}|^{|A|} & \longrightarrow & |\widetilde{A}| \times |\widetilde{A}|. \end{array}$$

Clearly, for every object $X \in \mathsf{X}$, the unit $\eta_X\colon X \to GF(X)$ is an isomorphism if and only if η_X is surjective and an embedding. If the dual adjunction is constructed using (Init X) and (Init A), then, by Remark 3.1.2,

η_X is an embedding if and only if the cone $(\psi\colon X \to \widetilde{X})_\psi$ is point-separating and initial.

We hasten to remark that the latter condition only depends on $\widetilde{X}$ and is independent of the choice of A. If η is not componentwise an embedding, we can replace X by its full subcategory defined by all those objects X where $(\psi\colon X \to \widetilde{X})_\psi$ is point-separating and initial; by construction, the functor G corestricts to this subcategory. Again, this procedure is only useful if this subcategory has all "interesting spaces", otherwise it is probably best to use a different dualising object. Getting back to adjunction (3.1.ii) we can easily conclude the proof of Stone's duality theorem. It is just a matter of observing that saying that X is a compact Hausdorff space such that the cone $(\psi\colon X \to 2)_\psi$ is point-separating and initial is just a different way of saying that X is a Boolean space. This also means that we cannot obtain a dual equivalence for the category of compact Hausdorff spaces in a natural way by starting with a dual adjunction based on the two-element set; to keep all the interesting objects in $\mathsf{Fix}(\mathsf{X})$ the two-element discrete space would need to be an initial cogenerator in the category $\mathsf{CompHaus}$. For exactly this reason, in Section 3.2 we will consider the compact Hausdorff space $[0, 1]$ instead of the discrete two-element space.

We assume now that η is componentwise an embedding. Then the functor $F\colon \mathsf{X} \to \mathsf{A}^{\mathrm{op}}$ is faithful, and η is an isomorphism if and only if F is also full. Put differently, if η is not an isomorphism, then A has too many arrows. A possible way to fix this problem is to enrich the structure of A. For instance, in [Johnstone, 1986, VI.4.4] it is shown that, under mild conditions, A can be substituted by the category of Eilenberg–Moore algebras for the monad on A induced by the dual of the adjunction (3.1.i). In this thesis we take a different approach: instead of saying "A has too many morphisms", one might also think that "X has too few morphisms". One way of adding morphisms to a category is to replace it by the Kleisli category of a suitable monad on it. In fact, and rather trivially, for the monad $\mathbb{T}$ induced by the adjunction (3.1.i), the comparison functor $\mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ is fully faithful. However, in general, this procedure will not be of practical interest since our knowledge about the monad induced by $F \dashv G$ might be very limited. The situation improves if we take a different, better known monad $\mathbb{T}$ on X *isomorphic* to the monad induced by $F \dashv G$. We are then left with the task of identifying the X-morphisms inside $\mathsf{X}_{\mathbb{T}}$ in a purely categorical way, to be translated across a duality.

Example 3.1.6. Consider the powerset monad $\mathbb{P}$ on Set whose Kleisli category $\mathsf{Set}_{\mathbb{P}}$ is equivalent to the category Rel of sets and relations. Within Rel, the following two fundamentally different properties identify functions.

- A relation is a function if and only if it has a right adjoint in the ordered category Rel. This is actually a 2-categorical property; if we want to use it in a duality we must make sure that the involved equivalence functors are locally monotone.

- A relation $r\colon X \nrightarrow Y$ is a function if and only if r is a homomorphism of comonoids in the monoidal category Rel, that is, the diagrams

  $$\begin{array}{ccc} X & \xrightarrow{\;r\;} & Y \\ & \searrow^{\top} & \downarrow{\scriptstyle \top} \\ & & 1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X \times X & \xrightarrow{\;r\times r\;} & Y \times Y \\ {\scriptstyle \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ X & \xrightarrow[\;r\;]{} & Y \end{array}$$

  commute. In the second diagram, $X \times X$ denotes the set-theoretical product which can be misleading since it is not the categorical product in Rel. To use this description in a duality result, one needs to know the corresponding operation on the other side.

In the next chapter we build up on the idea above to develop duality theory for categories containing all compact Hausdorff spaces. If we focused in the category BooSp, we could use Proposition 3.1.4 to quickly obtain the natural dual adjunction

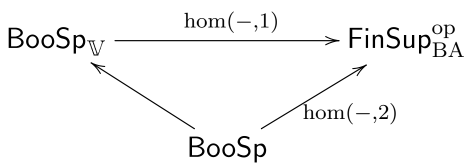
$$\mathsf{BooSp} \underset{\hom(-,2)}{\overset{\hom(-,2)}{\rightleftarrows}} \mathsf{FinSup}^{\mathrm{op}}_{\mathrm{BA}}. \qquad (3.1.\text{iv})$$

By starting with the adjunction above instead of the adjunction (3.1.ii) we ensure that every X-component of η is an embedding since the two-element discrete space is an initial cogenerator in BooSp. But now, the category BooSp "has too few morphisms". In the spirit of this work, we should replace it by the Kleisli category of a monad on BooSp. A suitable candidate is the classical Vietoris monad on BooSp (see Section 2.4). This choice leads us to sketch a proof of Halmos' duality theorem.

Theorem 3.1.7 ([Halmos, 1956])**.** *The Kleisli category of the Vietoris monad on* BooSp *(see Section 2.4) is dually equivalent to the category* $\mathsf{FinSup}_{\mathrm{BA}}$ *of Boolean algebras with finite suprema preserving maps.*

Proof. Observe that:

1. The functor $\hom(-,1)\colon \mathsf{BooSp}_{\mathbb{V}} \to \mathsf{FinSup}_{\mathrm{BA}}^{\mathrm{op}}$ extends the functor $\hom(-,2)\colon \mathsf{BooSp} \to \mathsf{FinSup}_{\mathrm{BA}}^{\mathrm{op}}$ so that the diagram below commutes.



2. The monad morphism induced by $\hom(-,1)$ is a natural isomorphism, therefore, by Theorem 2.1.11, the functor $\hom(-,1)$ is fully faithful.

3. Stone's representation theorem guarantees that the functor $\hom(-,1)$ is essentially surjective on objects.

□

To pass from Halmos' duality theorem to Stone's duality theorem we are left we the task of identifying the *relations* in $\mathsf{BooSp}_{\mathbb{V}}$ that are *functions* in a way that can translated across the duality. In [Hofmann and Nora, 2015] this is achieved using the description of relations as comonoids in the monoidal category Rel of Examples 3.1.6.

Theorem 3.1.8 ([Stone's dual equivalence]Stone [1936])**.** *The categories* BooSp *and* BA *are dually equivalent.*

In the considerations above, the Kleisli category $\mathsf{X}_{\mathbb{T}}$ was only introduced to support the study of X; however, at some occasions our primary interest lies in $\mathsf{X}_{\mathbb{T}}$ itself. In this case, a monad $\mathbb{T}$ on X is typically given before-hand, and we wish to find an adjunction (3.1.i) so that the induced monad is isomorphic to $\mathbb{T}$. If a dualising object $(\widetilde{X},\widetilde{A})$ induces this adjunction, we speak of a ***functional representation*** of $\mathbb{T}$. Looking again at the example $\mathrm{CoAlg}(V) \simeq \mathsf{BAO}^{\mathrm{op}}$ of Chapter 1, by observing that V is part of a monad $\mathbb{V} = (V, m, e)$ on BooSp, we can think of the objects of $\mathrm{CoAlg}(V)$ as Boolean spaces X equipped with an endomorphism $r\colon X \nrightarrow X$ in $\mathsf{BooSp}_{\mathbb{V}}$; the morphisms of $\mathrm{CoAlg}(V)$ are those morphisms of BooSp commuting with this additional structure. The duality $\mathrm{CoAlg}(V) \simeq \mathsf{BAO}^{\mathrm{op}}$ follows now from both Halmos' duality and the classical Stone duality $\mathsf{BooSp} \simeq \mathsf{BA}^{\mathrm{op}}$ [Stone, 1936].

As in the proof of Halmos duality theorem above, our aim is to construct and analyse functors $F\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ which extend a given functor $F\colon \mathsf{X} \to \mathsf{A}^{\mathrm{op}}$ that is part of an adjunction $F \dashv G$ induced by a dualising object $(\widetilde{X},\widetilde{A})$. It is well-known that such functors $F\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$

correspond precisely to monad morphisms from $\mathbb{T}$ to the monad induced by $F \dashv G$, and that monad morphisms into a "double dualisation monad" are in bijection with certain algebra structures on $\widetilde{X}$ (see [Kock, 1971], for instance). In the remainder of this section, we explain these correspondences in the specific context of this thesis.

Let X and A be categories with representable faithful functors

$$|-| \simeq \hom(X_0,-)\colon \mathsf{X} \longrightarrow \mathsf{Set} \qquad\text{and}\qquad |-| \simeq \hom(A_0,-)\colon \mathsf{A} \longrightarrow \mathsf{Set},$$

$\mathbb{T} = (T, m, e)$ a monad on X and $F \dashv G$ an adjunction

$$\mathsf{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{A}^{\mathrm{op}} \quad (\bot)$$

induced by $(\widetilde{X}, \widetilde{A})$. We denote by $\mathbb{D}$ the monad induced by $F \dashv G$. The next result establishes a connection between monad morphisms $j\colon \mathbb{T} \to \mathbb{D}$ and $\mathbb{T}$-algebra structures on $\widetilde{X}$ *compatible* with the adjunction $F \dashv G$.

Theorem 3.1.9. *In the setting described above, the following data are in bijection.*

1. *Monad morphisms* $j\colon \mathbb{T} \to \mathbb{D}$.
2. *Functors* $F\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ *making the diagram*

$$\begin{array}{ccc} \mathsf{X}_{\mathbb{T}} & \xrightarrow{F} & \mathsf{A}^{\mathrm{op}} \\ {\scriptstyle F_{\mathbb{T}}}\uparrow & \nearrow_{F} & \\ \mathsf{X} & & \end{array}$$

   commutative.

3. $\mathbb{T}$*-algebra structures* $\sigma\colon T\widetilde{X} \to \widetilde{X}$ *such that the map*

$$\hom(X, \widetilde{X}) \longrightarrow \hom(TX, \widetilde{X}),\ \psi \longmapsto \sigma \cdot T\psi$$

   is an A*-morphism* $\kappa_X\colon FX \to FTX$*, for every object* X *in* X.

Proof. The equivalence between the data described in (1) and (2) is well-known, see [Pumplün, 1970], for instance. We recall here that, for a monad morphism $j\colon \mathbb{T} \to \mathbb{D}$, the corresponding functor $F\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ can be obtained as

$$\mathsf{X}_{\mathbb{T}} \xrightarrow{\text{composition with } j} \mathsf{X}_{\mathbb{D}} \xrightarrow{\text{comparison}} \mathsf{A}^{\mathrm{op}}.$$

To describe the passage from (1) to (3), we recall from [Johnstone, 1986, Lemma VI.4.4] that

$\widetilde{X}$ becomes a $\mathbb{D}$-algebra since $\widetilde{X} \simeq GA_0$ and $G\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{X}$ factors as

$$\begin{array}{ccc} \mathsf{A}^{\mathrm{op}} & \xrightarrow{\text{comparison}} & \mathsf{X}^{\mathbb{D}} \\ & {\scriptstyle G}\searrow & \downarrow{\scriptstyle \text{forgetful}} \\ & & \mathsf{X}. \end{array}$$

A little computation shows that the $\mathbb{D}$-algebra structure on $\widetilde{X}$ is

$$GF\widetilde{X} \xrightarrow{\mathrm{ev}_{1_{\widetilde{X}}}} \widetilde{X}.$$

Composing $\mathrm{ev}_{1_{\widetilde{X}}}$ with $j_{\widetilde{X}}$ gives a $\mathbb{T}$-algebra structure $\sigma\colon T\widetilde{X} \to \widetilde{X}$. Furthermore, the functor $F\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ sends $1_{TX}\colon TX \nrightarrow X$ in $\mathsf{X}_{\mathbb{T}}$ to the A-morphism $Fj_X \cdot \varepsilon_{FX}\colon FX \to FTX$ which sends $\psi \in FX$ to $Fj_X(\mathrm{ev}_\psi) = \mathrm{ev}_\psi \cdot j_X$. On the other hand,

$$\sigma \cdot T\psi = \mathrm{ev}_{1_{\widetilde{X}}} \cdot j_{\widetilde{X}} \cdot T\psi = \mathrm{ev}_{1_{\widetilde{X}}} \cdot GF\psi \cdot j_X = \mathrm{ev}_\psi \cdot j_X;$$

which shows that $\kappa_X = Fj_X \cdot \varepsilon_{FX}$ is an A-morphism. For a compatible $\mathbb{T}$-algebra structures $\sigma\colon T\widetilde{X} \to \widetilde{X}$ as in (3),

$$(\varphi\colon X \to TY) \longmapsto (FY \xrightarrow{\kappa_Y} FTY \xrightarrow{F\varphi} FX)$$

defines a functor $F\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ making the diagram

$$\begin{array}{ccc} \mathsf{X}_{\mathbb{T}} & \xrightarrow{F} & \mathsf{A}^{\mathrm{op}} \\ {\scriptstyle F_{\mathbb{T}}}\uparrow & \nearrow{\scriptstyle F} & \\ \mathsf{X} & & \end{array}$$

commutative. The induced monad morphism $j\colon \mathbb{T} \to \mathbb{D}$ is given by the family of maps

$$j_X\colon |TX| \longrightarrow \hom(FX, \widetilde{A}),\ \mathfrak{x} \longmapsto (\psi \mapsto \sigma \cdot T\psi(\mathfrak{x})).$$

Furthermore, the $\mathbb{T}$-algebra structure induced by this j is indeed

$$\mathrm{ev}_{1_{\widetilde{X}}} \cdot j_{\widetilde{X}} = \sigma \cdot T1_{\widetilde{X}} = \sigma.$$

Finally, for a monad morphism $j\colon \mathbb{T} \to \mathbb{D}$, the monad morphism induced by the corresponding algebra structure σ has as X-component the map sending $\mathfrak{x} \in TX$ to

$$\sigma \cdot T\psi(\mathfrak{x}) = \mathrm{ev}_\psi \cdot j_X(\mathfrak{x}) = j_X(\mathfrak{x})(\psi). \qquad \square$$

Remark 3.1.10. The constructions described above seem to be more natural if $\widetilde{X} = TX_0$ with

$\mathbb{T}$-algebra structure m_{X_0}, see [Hofmann and Nora, 2015, Proposition 4.3]. In this case, the functor $F\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ is a lifting of the hom-functor $\hom(-, X_0)\colon \mathsf{X}_{\mathbb{T}} \to \mathsf{Set}^{\mathrm{op}}$. Furthermore, interpreting the elements of TX as morphisms $\varphi\colon X_0 \nrightarrow X$ in the Kleisli category $\mathsf{X}_{\mathbb{T}}$ allows to describe the components of the monad morphism j using composition in $\mathsf{X}_{\mathbb{T}}$:

$$j_X\colon |TX| \longrightarrow \hom(FX, \widetilde{A}),\ \varphi \longmapsto (\psi \mapsto \psi \cdot \varphi).$$

3.2 Enriched Halmos dualities

With the arguments of the previous section in mind, in this section we develop duality theory for separated ordered compact spaces. More specifically, we apply the results presented in Section 2.1 and Section 3.1 to the Vietoris monad (see Section 2.4) $\mathbb{V}$ on $\mathsf{X} = \mathsf{SepOrdComp}$, with $\widetilde{X} = [0,1]^{\mathrm{op}}$ and $\mathbb{V}$-algebra structure

$$V([0,1]^{\mathrm{op}}) \longrightarrow [0,1]^{\mathrm{op}},\ A \longmapsto \sup_{x \in A} x.$$

We begin by filling all the details that, for a category A unknown at the moment, lead to the construction of a commutative diagram

$$\begin{array}{ccc} \mathsf{SepOrdComp}_{\mathbb{V}} & \xrightarrow{\quad C \quad} & \mathsf{A}^{\mathrm{op}}, \\ & \nwarrow \qquad \nearrow{\scriptstyle C} & \\ & \mathsf{SepOrdComp} & \end{array}$$

where

$$\mathsf{SepOrdComp} \underset{G}{\overset{C}{\rightleftarrows}} \mathsf{A}^{\mathrm{op}} \quad (\bot)$$

is an adjunction induced by $([0,1]^{\mathrm{op}}, [0,1])$ compatible with the $\mathbb{V}$-algebra structure on $[0,1]^{\mathrm{op}}$. Denoting by $\mathbb{D}$ the monad induced by the adjunction above, it follows that the corresponding monad morphism $j\colon \mathbb{V} \to \mathbb{D}$ has as components the maps

$$j_X\colon VX \longrightarrow GC(X),\ A \longmapsto (\Phi_A\colon CX \to [0,1],\ \psi \mapsto \sup_{x \in A} \psi(x)).$$

In Section 3.2.1 we discuss how to turn j into an isomorphism. Finally, in Section 3.2.2, we use a Stone–Weierstraß type of theorem to obtain the duality results.

Assumption 3.2.1. From now on $\otimes$ is a quantale structure on $[0,1]$ with neutral element 1. Note that then necessarily $u \otimes v \leq u \wedge v$, for all $u, v \in [0,1]$. To combine continuous functions $\psi_1, \psi_2\colon X \to [0,1]$, we assume that $\otimes\colon [0,1] \times [0,1] \to [0,1]$ is continuous with respect to the Euclidean topology on $[0,1]$. In other words, we consider a continuous t-norm on $[0,1]$.

The first step to identify an appropriate category A is to introduce $[0,1]$-enriched notions analogous to the 2-enriched notions of distributive lattice and Boolean algebra appearing in the classical duality theorems of Stone and Halmos. A distributive lattice X is in particular a finite sup-lattice equipped with a commutative monoid structure $\wedge\colon X \times X \to X$ with neutral element the top-element of X and where, moreover, the maps $x \wedge - \colon X \to X$ preserve finite suprema. Also note that every monotone map $f\colon X \to Y$ between lattices laxly preserves infima, that is, for all $x, x' \in X$ the inequality $f(x \wedge x') \leq f(x) \wedge f(x')$ holds. By interpreting a finite sup-lattice as a finitely cocomplete 2-category, we can translate the description above naturally to the $[0,1]$-enriched setting. Below we introduce a $[0,1]$-enriched counterpart of distributive lattices where the monoid structure is not necessarily the infimum since the tensor product on $[0,1]$ need not be the infimum. We think of these $[0,1]$-categories as *generalised* $[0,1]$*-enriched (distributive) lattices.*

- The category
$$[0,1]\text{-}\mathsf{GLat}$$
has as objects separated finitely cocomplete $[0,1]$-categories X equipped with an associative and commutative operation $\circledcirc\colon X \times X \to X$ with unit element which is also the top-element of X and such that, for every $x \in X$, the map $x \circledcirc - \colon X \to X$ is a finitely cocontinuous $[0,1]$-functor; the morphisms of $[0,1]$-GLat are the finitely cocontinuous $[0,1]$-functors preserving the unit and the multiplication $\circledcirc$.

- The category
$$[0,1]\text{-}\mathsf{LaxGLat}$$
has the same objects as $[0,1]$-GLat; the morphisms are finitely cocontinuous $[0,1]$-functors $f\colon X \to Y$ preserving laxly the monoid structure, that is, for all $x, x' \in X$,
$$f(x \circledcirc x') \leq f(x) \circledcirc f(x')$$

If we had chosen the quantale 2 over a quantale in $[0,1]$ in the definitions above, we would recover every distributive lattice as an object of 2-GLat. However, not every object of 2-GLat comes from a distributive lattice.

Example 3.2.2. Every quantale in $[0,1]$ with neutral element 1 is an object of 2-GLat.

Remark 3.2.3. Every finitely cocomplete $\mathcal{V}$-category is copowered, and every copowered $\mathcal{V}$-category can be interpreted as an ordered set equipped with an action from $\mathcal{V}$ (for details, see Section 2.2). In this perspective, we can think of $\circledcirc\colon X \times X \to X$ as an "extension" of $\otimes\colon X \times [0,1] \to X$ and write $x \otimes x'$ instead of $x \circledcirc x'$. The reason is that for every $u \in [0,1]$ and $x \in X$, we have
$$x \circledcirc (1 \otimes u) = (x \circledcirc 1) \otimes u = x \otimes u.$$

In the remainder of the chapter we will use this description extensively.

We recall that $[0,1]$-FinSup denotes the category of separated finitely cocomplete $[0,1]$-categories and finite colimit preserving $[0,1]$-functors; the unit interval $[0,1]$ equipped with $\hom\colon [0,1]\times[0,1]\to[0,1]$ is an object of $[0,1]$-FinSup.

Remark 3.2.4. Thinking more in algebraic terms, $[0,1]$-GLat is a $\aleph_1$-ary quasivariety; in fact, by adding to the algebraic theory of $[0,1]$-FinSup (see Remark 2.2.11) the operations and equations describing the monoid structure, we obtain a presentation by operations and implications. In particular, this means that $[0,1]$-GLat is complete and cocomplete.

Regarding limits, the following result can be verified by routine calculation.

Proposition 3.2.5. *The forgetful functors*

$$[0,1]\text{-}\mathsf{GLat}\longrightarrow[0,1]\text{-}\mathsf{FinSup}\qquad\qquad [0,1]\text{-}\mathsf{GLat}\longrightarrow[0,1]\text{-}\mathsf{LaxGLat}$$

preserve limits.

The next step in our argument is to construct a dual adjunction linking the categories $\mathsf{SepOrdComp}$ and $[0,1]$-GLat. The starting point to do so is the well-known fact that $(\mathcal{V},\hom)$ is a $\mathcal{V}$-category. In the sequel we consider the $[0,1]$-category $[0,1]$ as an object of $[0,1]$-GLat with multiplication given by the tensor product $\otimes\colon[0,1]\times[0,1]\to[0,1]$ of $[0,1]$.

Proposition 3.2.6. *The dualising object* $([0,1]^{\mathrm{op}},[0,1])$ *induces a natural dual adjunction*

$$\mathsf{SepOrdComp}\ \underset{G}{\overset{C}{\rightleftarrows}}\ [0,1]\text{-}\mathsf{LaxGLat}^{\mathrm{op}}. \quad (\bot)$$

Here CX is given by $\mathsf{SepOrdComp}(X,[0,1]^{\mathrm{op}})$ *with all operations defined pointwise, and GA is the space* $[0,1]$-$\mathsf{LaxGLat}(A,[0,1])$ *equipped with the initial topology with respect to all evaluation maps*

$$\mathrm{ev}_a\colon [0,1]\text{-}\mathsf{LaxGLat}(A,[0,1])\longrightarrow[0,1]^{\mathrm{op}},\ \Phi\longmapsto\Phi(a).$$

Proof. In terms of the algebraic presentation of the $[0,1]$-category $[0,1]$ of Remark 2.2.11, the operations $\vee$ and $-\otimes u$ are morphisms $\vee\colon[0,1]^{\mathrm{op}}\times[0,1]^{\mathrm{op}}\to[0,1]^{\mathrm{op}}$ and $-\otimes u\colon[0,1]^{\mathrm{op}}\to[0,1]^{\mathrm{op}}$ in $\mathsf{SepOrdComp}$, and the order relation of $[0,1]^{\mathrm{op}}$ is closed in $[0,1]^{\mathrm{op}}\times[0,1]^{\mathrm{op}}$. Furthermore, $\otimes\colon[0,1]^{\mathrm{op}}\times[0,1]^{\mathrm{op}}\to[0,1]^{\mathrm{op}}$ is a morphism in $\mathsf{SepOrdComp}$. Therefore, the assertion follows from Theorem 3.1.3, Proposition 3.1.4 and Remark 3.1.5. □

Before proceeding we need to identify the $\mathbb{V}$-algebra structure of the separated ordered compact space $[0,1]^{\mathrm{op}}$.

Proposition 3.2.7. *The separated ordered compact space* $[0,1]^{\mathrm{op}}$ *is a* $\mathbb{V}$*-algebra with algebra structure* $\sup\colon V([0,1]^{\mathrm{op}}) \to [0,1]^{\mathrm{op}}$. *Furthermore, the function*

$$\hom(X,[0,1]^{\mathrm{op}}) \longrightarrow \hom(VX,[0,1]^{\mathrm{op}}),\ \psi \longmapsto (A \mapsto \sup_{x\in A}\psi(x))$$

is a morphism $CX \to CVX$ *in* $[0,1]$-$\mathsf{LaxGLat}$.

Finally, by Theorem 3.1.9 and Remark 3.1.10, we arrive at the commutative diagram of functors

$$\begin{array}{ccc} \mathsf{SepOrdComp}_{\mathbb{V}} & \xrightarrow{\quad C \quad} & [0,1]\text{-}\mathsf{LaxGLat}^{\mathrm{op}}; \\ & \nwarrow \qquad \nearrow_{C} & \\ & \mathsf{SepOrdComp} & \end{array}$$

where, for $\varphi\colon X \to\!\!\!\!\!\circ\ Y$ in $\mathsf{SepOrdComp}_{\mathcal{V}}$,

$$\begin{aligned} C\varphi\colon CY &\longrightarrow CX \\ \psi &\longmapsto \left(x \mapsto \sup_{x\,\varphi\,y}\psi(y)\right). \end{aligned}$$

In the next section we discuss how to turn the monad morphism $j\colon \mathbb{V} \to \mathbb{D}$ induced by $C\colon \mathsf{SepOrdComp}_{\mathbb{V}} \to [0,1]\text{-}\mathsf{LaxGLat}^{\mathrm{op}}$ into an isomorphism.

Proposition 3.2.8. *The monad morphism* $j\colon \mathbb{V} \to \mathbb{D}$ *is defined by the family of maps*

$$j_X\colon VX \longrightarrow [0,1]\text{-}\mathsf{LaxGLat}(CX,[0,1]),\ A \longmapsto \Phi_A,$$

with

$$\Phi_A\colon CX \longrightarrow [0,1],\ \psi \longmapsto \sup_{x\in A}\psi(x).$$

Proof. See the proof of Theorem 3.1.9. □

3.2.1 The natural transformation $j\colon \mathbb{V} \to \mathbb{D}$ is an isomorphism

Our first inspiration to turn j into an isomorphism stems from [Shapiro, 1992] where the following result is proven.

Theorem 3.2.9. *Consider the subfunctor* $V_1\colon \mathsf{CompHaus} \to \mathsf{CompHaus}$ *of* V *sending* X *to the space of all non-empty closed subsets of* X. *The functor* $V_1\colon \mathsf{CompHaus} \to \mathsf{CompHaus}$ *is naturally isomorphic to the functor which sends* X *to the space of all functions*

$$\Phi\colon C(X,\mathbb{R}_0^+) \longrightarrow \mathbb{R}_0^+$$

that, for all $\psi,\psi_1,\psi_2 \in C(X,\mathbb{R}_0^+)$ *and* $u \in \mathbb{R}_0^+$, *satisfy the conditions:*

1. Φ *is monotone,*

2. $\Phi(u * \psi) = u * \Phi(\psi)$,

3. $\Phi(\psi_1 + \psi_2) \leq \Phi(\psi_1) + \Phi(\psi_2)$,

4. $\Phi(\psi_1 \cdot \psi_2) \leq \Phi(\psi_1) \cdot \Phi(\psi_2)$,

5. $\Phi(\psi_1 + u) = \Phi(\psi_1) + u$,

6. $\Phi(u) = u$.

The topology on the set of all maps $\Phi\colon C(X, \mathbb{R}_0^+) \to \mathbb{R}_0^+$ *satisfying the conditions above is the initial one with respect to all evaluation maps* ev_ψ*, where* $\psi \in C(X, \mathbb{R}_0^+)$*. The* X*-component of the natural isomorphism sends a closed non-empty subset* $A \subseteq X$ *to the map* $\Phi_A\colon C(X, \mathbb{R}_0^+) \to \mathbb{R}_0^+$ *defined by*

$$\Phi_A(\psi) = \sup_{x \in A} \psi(x).$$

Shapiro's result shows that the subfunctor V_1 is isomorphic to a functor that resembles the one that we get from the adjunction of Proposition 3.2.6. Of course, to fit better into our framework, in the sequel we will consider functions into $[0,1]$ instead of $\mathbb{R}_0^+$, and binary suprema $\vee$ instead of $+$ in (3). The empty space is excluded in the definition of V_1, and at first sight it seems that allowing it in the definition of V might be problematic. After all, it is immediate to see that for $A = \varnothing$, the map Φ_A does not satisfy the last two axioms above. However, as we will see later, the condition (5) is not necessary for Shapiro's result; moreover, thanks to (2), the condition (6) can be equivalently expressed as $\Phi(1) = 1$, and this is purely related to $A \neq \varnothing$ (see Proposition 3.2.17). Therefore, the case $A = \varnothing$ is not problematic at all. Finally, condition (2) hints that Shapiro's formulation is consistent with the interpretation of copowered $[0,1]$-categories as ordered sets equipped with an action from $[0,1]$. In the sequel we follow this perspective and, like Shapiro, we also refer individually to the components of the structure of CX. In particular, we consider the following conditions on a map $\Phi\colon CX \to [0,1]$.

(Mon) Φ is monotone.

(Act) For every $u \in [0,1]$ and $\psi \in CX$, $\Phi(u \otimes \psi) = u \otimes \Phi(\psi)$.

(Sup) For every $\psi_1, \psi_2 \in CX$, $\Phi(\psi_1 \vee \psi_2) = \Phi(\psi_1) \vee \Phi(\psi_2)$.

(Ten)$_{\mathrm{lax}}$ For every $\psi_1, \psi_2 \in CX$, $\Phi(\psi_1 \otimes \psi_2) \leq \Phi(\psi_1) \otimes \Phi(\psi_2)$.

(Ten) For every $\psi_1, \psi_2 \in CX$, $\Phi(\psi_1 \otimes \psi_2) = \Phi(\psi_1) \otimes \Phi(\psi_2)$.

(Top) $\Phi(1) = 1$.

Remark 3.2.10. 1. The condition (Act) implies $\Phi(0) = 0$ and (Sup) and implies (Mon). Also note that, by (Mon) and (Act), if, for every $x \in X$, $\psi(x) \leq u$ then $\Phi(\psi) \leq u$. Finally, if $\otimes = \wedge$, then $(\text{Ten})_{\text{lax}}$ is a consequence of (Mon).

2. A function $\Phi\colon CX \to [0,1]$ is a morphism in $[0,1]$-$\mathsf{LaxGLat}$ if and only if satisfies (Mon), (Act), (Sup) and $(\text{Ten})_{\text{lax}}$and is a morphism in $[0,1]$-GLat if and only if satisfies the conditions(Mon), (Act), (Sup), (Top) and (Ten).

Our next goal is to invert the process $A \mapsto \Phi_A$. Firstly, following [Shapiro, 1992], we introduce the subsequent notation.

- For every map $\psi\colon X \to [0,1]$, $\mathcal{Z}(\psi) = \{x \in X \mid \psi(x) = 0\}$ denotes the zero-set of ψ. If ψ is a monotone and continuous map $\psi\colon X \to [0,1]^{\text{op}}$, then $\mathcal{Z}(\psi)$ is an closed upper subset of X.

- For every map $\Phi\colon CX \to [0,1]$, we put

$$\mathcal{Z}(\Phi) = \bigcap\{\mathcal{Z}(\psi) \mid \psi \in CX,\, \Phi(\psi) = 0\}.$$

  Note that $\mathcal{Z}(\Phi)$ is a closed upper subset of X.

There is arguably a more natural candidate for an inverse of j_X. Note that, given a set $\{A_i \mid i \in I\}$ of closed upper subsets of X with $A = \overline{\bigcup_{i \in I} A_i}$, for every $\psi \in CX$ one verifies

$$\Phi_A(\psi) = \sup_{x \in \bigcup_{i \in I} A_i} \psi(x) = \sup_{i \in I} \Phi_{A_i}(\psi).$$

Hence, the monotone map j_X preserves infima; [1] therefore it has a left adjoint which sends a morphism $\Phi\colon CX \to [0,1]$ to

$$\mathcal{A}(\Phi) = \bigcap_{\psi \in CX} \psi^{-1}[0, \Phi(\psi)].$$

In the sequel it will be convenient to consider the maps $\mathcal{Z}$ and $\mathcal{A}$ defined on the set of all maps from CX to $[0,1]$. We have the following elementary properties.

Lemma 3.2.11. *Let X be a separated ordered compact space X.*

1. *The maps $\mathcal{A}, \mathcal{Z}\colon \{\Phi\colon CX \to [0,1]\} \to VX$ are monotone.*

2. *For every map $\Phi\colon CX \to [0,1]$, $\mathcal{A}(\Phi) \subseteq \mathcal{Z}(\Phi)$.*

3. *For every $A \in VX$, $\mathcal{Z} \cdot j_X(A)) = A = \mathcal{A} \cdot j_X(A))$.*

4. *For every map $\Phi\colon CX \to [0,1]$ and every $\psi \in CX$, $j_X \cdot \mathcal{A}(\Phi)(\psi) \leq \Phi(\psi)$.*

[1] Note that the order is reversed.

Corollary 3.2.12. *For every separated ordered compact space X, the map $j_X \colon VX \to GCX$ is an order-embedding.*

Now, we discuss conditions to impose on the functions $\Phi \colon CX \to [0,1]$ so that j_X restricts to a bijection between VX and the subset of $\{\Phi \colon CX \to [0,1]\}$ defined by them. The conditions (Mon) and (Sup) hint that for a given value it would be convenient to find a ψ that in a subset of our space takes values as high as we want while keeping $\Phi(\psi)$ close enough to the given value. With the definitions of $\mathcal{Z}$ and $\mathcal{A}$ in mind, we consider:

(A) For every $x \in X$ and every $\psi \in CX$, if $\psi(x) > \Phi(\psi) = 0$, then there exists some $\bar{\psi} \in CX$ with $\bar{\psi}(x) = 1$ and $\Phi(\bar{\psi}) = 0$.

Lemma 3.2.13. *Let X be a separated ordered compact space.*

1. *If $\Phi \colon CX \to [0,1]$ satisfies (Mon), (Act) and (Ten)$_{lax}$, then Φ satisfies (A).*
2. *If the quantale $[0,1]$ does not have nilpotent elements and $\Phi \colon CX \to [0,1]$ satisfies (Mon) and (Act), then Φ satisfies (A).*

Proof. Assume $\psi(x) > \Phi(\psi) = 0$. Put $v = \psi(x)$ and take u with $0 < u < v$. Put $A = \psi^{-1}([0,u])$. By Proposition 2.3.9, there is some $\psi' \in CX$ with $A \subseteq \mathcal{Z}(\psi')$ and $\psi'(x) = 1$. Furthermore,

$$u \otimes \psi' \leq u \wedge \psi' \leq \psi$$

and therefore $u \otimes \Phi(\psi') \leq \Phi(\psi) = 0$. Since $u \neq 0$, we get $\Phi(\psi')^n = 0$ for some $n \in \mathbb{N}$. If there are no nilpotent elements, then $\Phi(\psi') = 0$. In general, using condition (Ten)$_{\text{lax}}$ we obtain $\Phi(\psi'^n) \leq \Phi(\psi')^n = 0$ and $\psi'^n(x) = 1$. □

In the next result we show that if $\Phi \colon CX \to [0,1]$ satisfies (Mon), (Sup) and (A) then for every $\psi \in CX$, the inequality $\Phi(\psi) \leq \sup_{x \in \mathcal{Z}(\Phi)} \psi(x)$ holds. For a finite space we could prove this by using (A) to find functions $\psi_1, \dots, \psi_n \in CX$ such that $\psi \leq \psi_1 \otimes \psi \vee \cdots \vee \psi_n \otimes \psi$ and $\Phi(\psi_i) \leq \sup_{x \in \mathcal{Z}(\Phi)} \psi(x)$. Then, the result would follow by applying (Mon) and (Sup). The core idea behind the proof of the general case is the same, however, the passage from the finite to the infinite case poses additional technical challenges.

Proposition 3.2.14. *Let X be a separated ordered compact space. For every $\Phi \colon CX \to [0,1]$ satisfying (Mon), (Act), (Sup) and (A),*

$$\Phi(\psi) \leq j_X \cdot \mathcal{Z}(\Phi)(\psi),$$

for all $\psi \in CX$.

Proof. Let $\psi \in CX$, we wish to show that $\Phi(\psi) \leq \sup_{x \in Z(\Phi)} \psi(x)$. To this end, consider an element $u \in [0,1]$ with $\sup_{x \in Z(\Phi)} \psi(x) < u$. Put

$$U = \{x \in X \mid \psi(x) < u\}.$$

Clearly, U is open and $Z(\Phi) \subseteq U$. Let now $x \in X \setminus Z(\Phi)$. There is some $\psi' \in CX$ with $\Phi(\psi') = 0$ and $\psi'(x) \neq 0$; by (A) we may assume $\psi'(x) = 1$. Let now $\alpha < 1$. For every $\psi' \in CX$ we put

$$\mathrm{supp}_\alpha(\psi') = \{x \in X \mid \psi'(x) > \alpha\}.$$

By the considerations above,

$$X = U \cup \bigcup \{\mathrm{supp}_\alpha(\psi') \mid \psi' \in C(X), \Phi(\psi') = 0\};$$

since X is compact, we find $\psi_1, \dots, \psi_n$ with $\Phi(\psi_i) = 0$ and

$$X = U \cup \mathrm{supp}_\alpha(\psi_1) \cup \dots \cup \mathrm{supp}_\alpha(\psi_n).$$

Hence,

$$\alpha \otimes \psi \leq u \vee (\psi_1 \otimes \psi) \vee \dots \vee (\psi_n \otimes \psi),$$

and therefore

$$\alpha \otimes \Phi(\psi) \leq u \vee \Phi(\psi_1 \otimes \psi) \vee \dots \vee \Phi(\psi_n \otimes \psi) \leq u \vee \Phi(\psi_1) \vee \dots \Phi(\psi_n) = u. \qquad \square$$

Hence, under the conditions of the proposition above, for every $\psi \in CX$, we have

$$\sup_{x \in \mathcal{A}(\Phi)} \psi(x) \leq \Phi(\psi) \leq \sup_{x \in \mathcal{Z}(\Phi)} \psi(x).$$

Now we look for conditions that guarantee that the equality $\mathcal{Z}(\Phi) = \mathcal{A}(\Phi)$ holds.

Proposition 3.2.15. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. If Φ satisfies (Mon), (Act) and (Ten)$_{lax}$ then $\mathcal{Z}(\Phi) = \mathcal{A}(\Phi)$.*

Proof. We consider first $\otimes = *$, in this case the proof is essentially taken from [Shapiro, 1992]. For every $\psi \in CX$ and every open lower subset $U \subseteq X$ with $U \cap Z(\Phi) \neq \varnothing$, we show that $\inf_{x \in U} \psi(x) \leq \Phi(\psi)$. To see this, put $u = \inf_{x \in U} \psi(x)$. Since there exists $z \in U \cap Z(\Phi)$, there is some $\psi' \in CX$ with $U^{\complement} \subseteq Z(\psi')$ and $\psi'(z) = 1$; thus $\Phi(\psi') \neq 0$. Then $u * \psi' \leq \psi * \psi'$ and therefore $u * \Phi(\psi') \leq \Phi(\psi) * \Phi(\psi')$. Since $\Phi(\psi') \neq 0$, we obtain $u \leq \Phi(\psi)$.

Let $x \in \mathcal{Z}(\Phi)$, $\psi \in CX$ and $v > \Phi(\psi)$. Put $U = \{x \in X \mid \psi(x) > v\}$. By the discussion above, $U \cap \mathcal{Z}(\Phi) = \varnothing$, hence $\psi(x) \leq v$. Therefore we conclude that $x \in \mathcal{A}(\Phi)$.

Consider now $\otimes = \odot$. Let $x \notin \mathcal{A}(\Phi)$. Then, there is some $\psi \in CX$ with $\psi(x) > \Phi(\psi)$.

With $u = \psi(x)$, we obtain

$$\hom(u, \psi(x)) = 1 > \hom(u, \Phi(\psi)) = u \pitchfork \Phi(\psi) \geq \Phi(u \pitchfork \psi),$$

using Remark 2.2.15 and that $\hom(u, -)\colon [0,1] \to [0,1]$ is monotone and continuous. Therefore we may assume that $\psi(x) = 1$. Since $\Phi(\psi) < 1$, there is some $n \in \mathbb{N}$ with $\Phi(\psi)^n = 0$, hence $\psi^n(x) = 1$ and $\Phi(\psi^n) = 0$. We conclude that $x \notin \mathcal{Z}(\Phi)$. □

From the results above we obtain a $[0,1]$-enriched counterpart of step 2 of the proof of Halmos' duality theorem (see Theorem 3.1.7).

Theorem 3.2.16. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Then the monad morphism j between the monad $\mathbb{V}$ on* SepOrdComp *and the monad induced by the adjunction $C \dashv G$ of Proposition 3.2.6 is an isomorphism. Therefore the functor*

$$C\colon \mathsf{SepOrdComp}_{\mathbb{V}} \longrightarrow [0,1]\text{-}\mathsf{LaxGLat}^{\mathrm{op}}$$

is fully faithful.

For $\Phi\colon CY \to CX$ in $[0,1]$-LaxGLat, the corresponding distributor $\varphi\colon X \nrightarrow Y$ is given by

$$x \,\varphi\, y \iff y \in \bigcap_{\Phi(\psi)(x)=0} Z(\psi).$$

As described in Section 3.1, to pass from Halmos' duality theorem to Stone's duality theorem we need to identify the functions among the relations in $\mathsf{BooSp}_{\mathbb{V}}$.

Proposition 3.2.17. *Let X be a separated ordered compact space and $A \subseteq X$ a closed upper subset of X. The following assertions hold.*

1. *$A \neq \varnothing$ if and only if Φ_A satisfies (Top).*
2. *A is irreducible as a subset of the corresponding stably compact space of X if and only if Φ_A satisfies (Ten).*

Proof. (1) is clear, and so is the implication " $\implies$ " in (2). Assume now that Φ_A satisfies (Ten) and let $A_1, A_2 \subseteq X$ be closed upper subsets with $A_1 \cup A_2 = A$. Let $x \notin A_1$ and $y \notin A_2$. We find $\psi_1, \psi_2 \in CX$ with

$$\psi_1(x) = 1, \qquad \psi_2(y) = 1, \qquad \forall z \in A \,.\, \psi_1(z) = 0 \text{ or } \psi_2(z) = 0.$$

Therefore

$$0 = \Phi_A(\psi_1 \otimes \psi_2) = \Phi_A(\psi_1) \otimes \Phi_A(\psi_2).$$

By Corollary 2.2.21, $\Phi_A(\psi_1) = 0$ or, for some $n \in \mathbb{N}$, $\Phi_A(\psi_2^n) = \Phi_A(\psi_2)^n = 0$, hence $x \notin A$ or $y \notin A$. We conclude that $A = A_1$ or $A = A_2$. □

We recall from Example 3.1.6 that a relation is a function if and only if it is a comonoid in the monoidal category Rel.

Corollary 3.2.18. *Let* $\varphi\colon X \nrightarrow Y$ *in* $\mathsf{SepOrdComp}_{\mathbb{V}}$*. Then:*

1. φ *is a total relation if and only if* $C\varphi$ *preserves* 1.
2. φ *is a partial function if and only if* $C\varphi$ *preserves* $\otimes$.

From Corollary 3.2.18 we obtain

Corollary 3.2.19. *Assume that* $\otimes = *$ *is the multiplication or* $\otimes = \odot$ *is the Łukasiewicz tensor. Then the functor*

$$C\colon \mathsf{SepOrdComp} \longrightarrow [0,1]\text{-}\mathsf{GLat}^{\mathrm{op}}$$

is fully faithful.

The following examples show that Theorem 3.2.16 and Corollary 3.2.19 cannot be generalised to arbitrary continuous quantale structures on $[0,1]$; not even if, in the case of Theorem 3.2.16, we restrict $\mathsf{SepOrdComp}_{\mathbb{V}}$ to the full subcategory $\mathsf{CompHaus}_{\mathbb{V}}$. However, in Theorem 3.2.27 we show that Corollary 3.2.19 still holds if we restrict $\mathsf{SepOrdComp}$ to the full subcategory $\mathsf{CompHaus}$.

Examples 3.2.20. Consider $\otimes = \wedge$.

- For $X = 1$, the set $V1$ contains two elements; however, for every $\alpha \in [0,1]$, the map $\Phi = \alpha \wedge -\colon [0,1] \to [0,1]$ satisfies (Mon), (Act), (Sup) and $(\mathrm{Ten})_{\mathrm{lax}}$.
- For the compact Hausdorff space $X = \{0,1\}$, the set VX contains four elements; however, for every $\alpha \in [0,1]$, the map

  $$\Phi_\alpha\colon [0,1] \times [0,1] \longrightarrow [0,1],\ (u,v) \longmapsto u \vee (\alpha \wedge v)$$

  satisfies (Mon), (Act), (Sup) and $(\mathrm{Ten})_{\mathrm{lax}}$ (but, in general, not (Ten)); moreover, $\alpha = \Phi_\alpha(0,1)$ and therefore $\Phi_\alpha \neq \Phi_\beta$ for $\alpha \neq \beta$.
- For the separated ordered compact space $X = \{0 \geq 1\}$, $CX = \{(u,v) \in [0,1] \times [0,1] \mid u \leq v\}$ and VX contains three elements; however, for every $\alpha \in [0,1]$, the map

  $$\Phi_\alpha\colon CX \longrightarrow [0,1],\ (u,v) \longmapsto u \vee (\alpha \wedge v)$$

  satisfies (Mon), (Act), (Sup), (Ten) and (Top). In comparison with the previous example, the non-discrete order of X allows to show that Φ_α satisfies (Ten). To see this, take

$(u, v), (u', v') \in CX$. Then,

$$\begin{aligned}\Phi_\alpha(u,v) \wedge \Phi_\alpha(u',v') &= (u \wedge u') \vee (\alpha \wedge u \wedge v') \vee (\alpha \wedge v \wedge u') \vee (\alpha \wedge v \wedge v')\\ &= (u \wedge u') \vee (\alpha \wedge v \wedge v') = \Phi_\alpha\big((u,v) \wedge (u',v')\big).\end{aligned}$$

To deal with the general case, we introduce the following condition on a map $\Phi\colon CX \to [0,1]$ where $\ominus$ denotes truncated minus on $[0,1]$.

(Min) For every $u \in [0,1]$ and every $\psi \in CX$, $\Phi(\psi \ominus u) = \Phi(\psi) \ominus u$.

This condition is reminiscent of Shapiro's condition 5; however, contrary to what happens with 5, the condition (Min) is satisfied by $\Phi_A\colon CX \to [0,1]$ for every closed upper subset $A \subseteq X$. Clearly, for every closed upper subset $A \subseteq X$, the map $\Phi_A\colon CX \to [0,1]$ satisfies (Min).

Proposition 3.2.21. *Let X be a separated ordered compact space and $\Phi\colon CX \to [0,1]$ a map satisfying (Min). Then*

$$\mathcal{A}(\Phi) = \mathcal{Z}(\Phi).$$

Proof. Assume $x \notin \mathcal{A}(\Phi)$. Then there is some $\psi \in CX$ with $\psi(x) > \Phi(\psi)$. Put $u = \Phi(\psi)$. Then $\Phi(\psi \ominus u) = 0$ and $(\psi \ominus u)(x) > 0$, hence $x \notin \mathcal{Z}(\Phi)$. □

Therefore we obtain:

Proposition 3.2.22. *Let X be a separated ordered compact space. The map*

$$\begin{aligned} j_X\colon VX &\longrightarrow \{\Phi\colon CX \to [0,1] \mid \Phi \text{ satisfies (Mon), (Act), (Sup), (Ten)}_{lax} \text{ and (Min)}\}\\ A &\longmapsto \Phi_A\end{aligned}$$

is bijective. If the quantale $[0,1]$ does not have nilpotent elements, then j_X is bijective even if the condition (Ten)$_{lax}$ is dropped on the right hand side.

Accordingly, we introduce the categories

$$[0,1]\text{-}\mathsf{GLat}_\ominus \qquad\text{and}\qquad [0,1]\text{-}\mathsf{LaxGLat}_\ominus$$

defined as $[0,1]$-GLat and $[0,1]$-$\mathsf{LaxGLat}$ respectively, but the objects have an additional action $\ominus\colon X \times [0,1] \to X$ and the morphisms preserve it.

With the action $\ominus\colon [0,1] \times [0,1] \to [0,1]$, $(u,v) \mapsto u \ominus v$, the $[0,1]$-category $[0,1]$ is an object of both categories. As before (see Proposition 3.2.6 and Theorem 3.2.16), we obtain:

Theorem 3.2.23. *Under Assumption 3.2.1, the dualising object* $([0,1]^{\mathrm{op}}, [0,1])$ *induces a natural dual adjunction*

$$\mathsf{SepOrdComp} \underset{G}{\overset{C}{\rightleftarrows}} ([0,1]\text{-}\mathsf{LaxGLat}_{\ominus})^{\mathrm{op}}. \quad (C \perp G)$$

Here CX *is given by* $\mathsf{SepOrdComp}(X, [0,1]^{\mathrm{op}})$ *with all operations defined pointwise, and* GA *is the space* $[0,1]\text{-}\mathsf{LaxGLat}_{\ominus}(A, [0,1])$ *equipped with the initial topology with respect to all evaluation maps*

$$\mathrm{ev}_a \colon [0,1]\text{-}\mathsf{LaxGLat}_{\ominus}(A, [0,1]) \longrightarrow [0,1]^{\mathrm{op}},\ \Phi \longmapsto \Phi(a).$$

Furthermore, the following diagram of functors

$$\begin{array}{ccc} \mathsf{SepOrdComp}_{\mathbb{V}} & \xrightarrow{\quad C \quad} & ([0,1]\text{-}\mathsf{LaxGLat}_{\ominus})^{\mathrm{op}}, \\ & \nwarrow \qquad \nearrow_{C} & \\ & \mathsf{SepOrdComp} & \end{array}$$

commutes, and the induced monad morphism j *between* $\mathbb{V}$ *and the monad induced by* $C \dashv G$ *is an isomorphism. Therefore the functor*

$$C \colon \mathsf{SepOrdComp}_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathsf{LaxGLat}_{\ominus})^{\mathrm{op}}$$

is fully faithful, and so is the functor

$$C \colon \mathsf{SepOrdComp} \longrightarrow ([0,1]\text{-}\mathsf{GLat}_{\ominus})^{\mathrm{op}}.$$

Remark 3.2.24. Now that we know that $C \colon \mathsf{SepOrdComp} \to ([0,1]\text{-}\mathsf{GLat}_{\ominus})^{\mathrm{op}}$ is fully faithful, we can add further operations to the algebraic theory of $[0,1]\text{-}\mathsf{GLat}_{\ominus}$ if they can be transported pointwise from $[0,1]$ to CX. More precisely, let $\aleph$ be a cardinal and $h \colon [0,1]^{\aleph} \to [0,1]$ a monotone continuous map. If we add to the algebraic theory of $[0,1]\text{-}\mathsf{GLat}_{\ominus}$ an operation symbol of arity $\aleph$, then $C \colon \mathsf{SepOrdComp} \to ([0,1]\text{-}\mathsf{GLat}_{\ominus})^{\mathrm{op}}$ lifts to a fully faithful functor from $\mathsf{SepOrdComp}$ to the dual of the category of algebras for this theory by interpreting the new operation symbol in CX by

$$(f_i)_{i\in\aleph} \longmapsto (X \xrightarrow{\langle f_i\rangle_{i\in\aleph}} [0,1]^{\aleph} \xrightarrow{h} [0,1]).$$

Every $[0,1]\text{-}\mathsf{GLat}_{\ominus}$ morphism of type $CY \to CX$ preserves this new operation automatically. For instance, if $\hom(u,-) \colon [0,1] \to [0,1]$ is continuous, then CX has u-powers with $(\psi \pitchfork u)(x) = \hom(u, \psi(x))$, for all $x \in X$. Furthermore, every morphism $\Phi \colon CX \to CY$ in $[0,1]\text{-}\mathsf{GLat}_{\ominus}$ preserves u-powers.

In 1983, Banaschewski showed that CompHaus fully embeds into the category of distributive lattices equipped with constants from $[0,1]$ and constant preserving lattice homomorphisms. As we pointed out in Remark 2.2.8, instead of adding constants to the lattice CX of continuous $[0,1]$-valued functions, one could consider as well actions of type $u \wedge \psi$ of $[0,1]$ on the lattice CX. Therefore Banaschewski's result should be a special case of Theorem 3.2.16 for $\otimes = \wedge$. Unfortunately, this is not immediately the case since we need the additional operation $\ominus$. Still, using some arguments of [Banaschewski, 1983], we finish this section showing that for every compact Hausdorff space and every $\Phi\colon CX \to [0,1]$ in $[0,1]$-GLat, we have $\Phi_{Z(\Phi)} = \Phi$.

The next proposition is analogous to Proposition 3.2.17.

Proposition 3.2.25. *Let X be a separated ordered compact space and assume that $\Phi\colon CX \to [0,1]$ satisfies (Mon), (Act), (Sup)and (Ten)$_{lax}$.*

1. *If Φ satisfies also (Top), then $\mathcal{Z}(\Phi) \neq \varnothing$.*
2. *If Φ satisfies also (Ten), then $\mathcal{Z}(\Phi)$ is a irreducible subset of the corresponding stably compact space of X.*

Proof. To see the first implication: $1 = \Phi(1) \leq \sup_{x\in\mathcal{Z}(\Phi)} 1$, hence $\mathcal{Z}(\Phi) \neq \varnothing$. The proof of the second one is the same as the corresponding proof for Proposition 3.2.17. □

Lemma 3.2.26. *Let X be a compact Hausdorff space and $\Phi\colon CX \to [0,1]$ in $[0,1]$-GLat. We denote by x_0 the unique element of X with $\mathcal{Z}(\Phi) = \{x_0\}$. Then, for every $\psi \in CX$, $\psi(x_0) = \Phi(\psi)$.*

Proof. By Proposition 3.2.14, $\Phi(\psi) \leq \psi(x_0)$. To see the reverse inequality, let $u < \psi(x_0)$. Then $x_0 \notin \{x \in X \mid \psi(x) \leq u\}$, therefore there is some $\psi' \in CX$ with $\psi'(x_0) = 0$ and ψ' is constant 1 on $\{x \in X \mid \psi(x) \leq u\}$. Hence, $u \vee \psi' \leq \psi \vee \psi'$. Since $\Phi(\psi') \leq \psi'(x_0) = 0$, we conclude that $u = \Phi(u) \leq \Phi(\psi)$. □

Theorem 3.2.27. *Under Assumption 3.2.1, the functor*

$$C\colon \mathsf{CompHaus} \longrightarrow [0,1]\text{-}\mathsf{GLat}^{\mathrm{op}}$$

is fully faithful.

3.2.2 A Stone–Weierstraß theorem for $[0,1]$-categories

For a compact space X, the classic Stone–Weierstraß theorem (see [Stone, 1948a,b]) tells us that every subalgebra of the algebra $C(X,\mathbb{R})$ of continuous functions from X to $\mathbb{R}$ with enough elements to separate points is dense in $C(X,\mathbb{R})$. There are several possible formulations of this theorem according to the different algebraic structures that one wishes to consider.

In this section we present a version of the classical Stone–Weierstraß approximation theorem adapted to the context of $[0,1]$-categories to identify the image of the fully faithful functor

$$C\colon \mathsf{SepOrdComp}_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathsf{LaxGLat}_{\ominus})^{\mathrm{op}}.$$

The idea is to characterise the $[0,1]$-categories that are *dense* subcategories of the $[0,1]$-category CX. Then we can describe CX as a *closed* $[0,1]$-category that satisfies such characterisation.

Assumption. We continue working under Assumption 3.2.1.

We recall that for every separated ordered compact space X, the $[0,1]$-category CX is finitely cocomplete with $[0,1]$-category structure

$$d(\psi_1,\psi_2) = \inf_{x\in X} \hom(\psi_1(x),\psi_2(x)),$$

for all $\psi_1,\psi_2 \in CX$.

To define what it means for a $[0,1]$-category to be dense or closed we will use the closure operator for $\mathcal{V}$-categories introduced in Hofmann and Tholen [2010] (see Theorem 2.2.13). Applied in the $[0,1]$-category CX, this means, by Theorem 2.2.13, that a $\psi \in CX$ belongs to the closure of a subset $M \subseteq CX$ if and only if for every $0 \leq u < 1$, there is some $\psi' \in M$ with $u \leq d(\psi,\psi')$ and $u \leq d(\psi',\psi)$.

Similarly to the classical Stone–Weierstraß theorem, we also consider a separation condition on a subset L of CX that dictates how L should "look like" with respect to some pairs of points of X.

(Sep) for every $(x,y) \in X \times X$, with $x \not\geq y$, there is a $\psi \in L$ and an open neighbourhood U_y of y such that $\psi(x) = 1$ and, for all $z \in U_y$, $\psi(z) = 0$.

Lemma 3.2.28. *Let $L \subseteq CX$ be closed in CX under finite suprema, the monoid structure and the action of $[0,1]$; that is, for all $\psi_1,\psi_2 \in L$ and $u \in [0,1]$, $\psi_1 \vee \psi_2 \in L$, $\psi_1 \otimes \psi_2 \in L$, $1 \in L$ and $u \otimes \psi_1 \in L$. Let $\psi \in CX$. If the map* $\hom\colon \operatorname{im}(\psi) \times [0,1] \to [0,1]$ *is continuous and L satisfies (Sep), then $\psi \in \overline{L}$.*

Proof. Fix $x \in X$. Let $(\psi_y)_{y\in X}$ be the family of functions defined in the following way:

- if $y \not\leq x$, let ψ_y be a function guaranteed by (Sep) and U_y the corresponding neighbourhood;
- if $y \leq x$, then ψ_y is the constant function $\psi(x)$.

By hypothesis, the functions $\hom(\psi(x),-)\colon [0,1] \to [0,1]$ and ψ are continuous. Thus, the set

$$U_x = \{z \in X \mid u < \hom(\psi(x),\psi(z))\}$$

is an open neighbourhood of every $y \leq x$, and for such $y \in X$ we put $U_y = U_x$. Consequently, the collection of sets U_y ($y \in X$) is an open cover of X. By compactness of X, there exists a finite subcover $U_{y_1}, \dots, U_{y_n}, U_x$ of X. Considering the corresponding functions $\psi_{y_1}, \dots, \psi_{y_n}, \psi_x$, we define $\phi_x = \psi_{y_1} \otimes \cdots \otimes \psi_{y_n} \otimes \psi_x$.

By construction, ϕ_x has the following properties:

- $\phi_x(x) = \psi(x)$, since $\psi_{y_i}(x) = 1$ for $1 \leq i \leq n$ and $\psi_x(x) = \psi(x)$;
- for every $z \in X$, $u \otimes \phi_x(z) \leq \psi(z)$, since $z \in U_x$ or $z \in U_{y_i}$, for some i.

Now, for every $x \in X$ the set

$$V_x = \{z \in X \mid u < \hom(\psi(z), \phi_x(z))\}$$

is open because the functions $\hom\colon \operatorname{im}(\psi) \times [0,1] \to [0,1]$, ϕ_x and ψ are continuous. Therefore the collection of the sets V_x is an open cover of X. Again, by compactness of X, there exists a finite subcover $V_{x_1}, \dots, V_{x_m}$ of X. By defining $\phi = \phi_{x_1} \vee \cdots \vee \phi_{x_m}$ we obtain a function in L such that for every $z \in X$:

- $u \otimes \phi(z) = \displaystyle\bigvee_{j=1}^{m} u \otimes \phi_{x_j}(z) \leq \psi(z)$;
- $u \otimes \psi(z) \leq \phi(z)$. □

Remark 3.2.29. For the Łukasiewicz tensor, the lemma above affirms that L is dense in CX in the usual sense since, in this case,

$$\hom(u, v) \geq 1 - \varepsilon \iff \max(v - u, 0) \leq \varepsilon,$$

for all $u, v \in [0,1]$. However, if the tensor product is multiplication, the function $\hom\colon [0,1] \times [0,1] \to [0,1]$ is not continuous in $(0,0)$; as we will see in Lemma 3.2.31, to obtain a useful Stone-Weierstraß theorem this fact will require us to add a further condition involving truncated minus. Finally, if the tensor is the infimum, we cannot expect to obtain a useful approximation theorem using this closure. For example, for the separated ordered compact space $1 = \{*\}$ the topology in $CX \simeq [0,1]$ is generated by the sets $\{u\}$ and $]u, 1]$ with $u \neq 1$. For $x \neq 1$ and $M \subseteq [0,1]$, this means that the seemingly weaker condition $x \in \overline{M}$ actually implies that $x \in M$.

In light of Remark 3.2.29 above, in the remainder of the section we discuss the cases of the Łukasiewicz tensor and multiplication.

Lemma 3.2.30. *Let $\otimes = \odot$ be the Łukasiewicz tensor and $L \subseteq CX$. Assume that L is closed in CX under the monoid structure and u-powers, for all $u \in [0,1]$, and that the cone $(f\colon X \to [0,1]^{\mathrm{op}})_{f \in L}$ is initial; that is, for all $x, y \in X$, $x \geq y$ if and only if, for all $\psi \in L$, $\psi(x) \leq \psi(y)$. Then L satisfies (Sep).*

Proof. Let $(x, y) \in X \times X$ with $x \not\geq y$. By hyphotesis, there exists $\psi \in L$ and $c \in [0, 1]$ such that $\psi(x) > c > \psi(y)$. Let $u = \psi(x)$. Since L is closed for u-powers then $\psi' = \hom(u, \psi) \in L$. By Corollary 2.2.21 there exists $n \in \mathbb{N}$ such that $c^n = 0$. Therefore $\psi'^n(x) = 1$ and for all $z \in U_y = \psi^{-1}[0, c[$, $\psi'^n(z) = 0$. $\square$

Lemma 3.2.31. *Let $\otimes = *$ be the multiplication and $L \subseteq CX$. Assume that L is closed in CX under u-powers and $- \ominus u$, for all $u \in [0, 1]$, and that the cone $(f : X \to [0, 1]^{\mathrm{op}})_{f \in L}$ is initial. Then L satisfies (Sep).*

Proof. Let $(x, y) \in X \times X$ with $x \not\geq y$. By hyphotesis, there exists $\psi \in L$ and $c \in [0, 1]$ such that $\psi(x) > c > \psi(y)$. Let $\psi' = \psi \ominus c$ and $u = \psi'(x)$. Let $\psi'' = \hom(u, \psi') \in L$ and $U_y = \psi'^{-1}[0, c[$. Clearly, $\psi''(x) = 1$ and, since $u > 0$, for all $z \in U_y$ we obtain $\psi''(z) = 0$. $\square$

The results above tell us that certain $[0, 1]$-subcategories of CX are actually equal to CX if they are closed in CX. To ensure this property, we will work now with Cauchy-complete $[0, 1]$-categories. But first we need to make sure that the $[0, 1]$-category CX is Cauchy-complete.

Lemma 3.2.32. *The subset*

$$\{(u, v) \mid u \leq v\} \subseteq [0, 1] \times [0, 1]$$

of the $[0, 1]$-category $[0, 1] \times [0, 1]$ is closed.

Proof. Just observe that $\{(u, v) \mid u \leq v\}$ can be presented as the equaliser of the $[0, 1]$-functors $\wedge : [0, 1] \times [0, 1] \to [0, 1]$ and $\pi_1 : [0, 1] \times [0, 1] \to [0, 1]$. $\square$

Corollary 3.2.33. *For every separated ordered compact space X, the subset*

$$\mathsf{SepOrdComp}(X, [0, 1]^{\mathrm{op}}) \subseteq [0, 1]^{|X|}$$

of the $[0, 1]$-category $[0, 1]^{|X|}$ is closed.

With $U : \mathsf{Set} \to \mathsf{Set}$ denoting the ultrafilter functor, we write

$$\xi : U[0, 1] \longrightarrow [0, 1],\ \xi(\mathfrak{x}) = \sup_{A \in \mathfrak{x}} \inf_{u \in A} u = \inf_{A \in \mathfrak{x}} \sup_{u \in A} u.$$

for the convergence of the Euclidean topology of $[0, 1]$.

Lemma 3.2.34. *For every set X and every ultrafilter $\mathfrak{x}$ on X, the map*

$$\Phi_{\mathfrak{x}} : [0, 1]^X \longrightarrow [0, 1],\ \psi \longmapsto \xi \cdot U\psi(\mathfrak{x})$$

is a $[0, 1]$-functor.

Proof. Since domain and codomain of $\Phi_{\mathfrak{x}}$ are both $\mathcal{V}$-copowered, the assertion follows from

$$\xi \cdot U\psi \leq \xi \cdot U\psi' \qquad \text{and} \qquad \xi \cdot U(\psi \otimes u) = (\xi \cdot U\psi) \otimes u,$$

for all $u \in [0,1]$ and $\psi, \psi' \in [0,1]^X$ with $\psi \leq \psi'$. □

Corollary 3.2.35. *For every compact Hausdorff space X, the subset*

$$\mathsf{CompHaus}(X, [0,1]) \subseteq [0,1]^{|X|}$$

of the $[0,1]$-category $[0,1]^{|X|}$ is closed.

Proof. For an ultrafilter $\mathfrak{x} \in UX$ with convergence point $x \in X$, a map $\psi \colon X \to [0,1]$ preserves this convergence if and only if ψ belongs to the equaliser of $\Phi_{\mathfrak{x}}$ and π_x. □

Proposition 3.2.36. *For every separated ordered compact space X, the $[0,1]$-category CX is Cauchy-complete.*

We will now introduce a category A of $[0,1]$-categories which depends on the chosen tensor $\otimes$ on $[0,1]$.

For the Łukasiewicz tensor $\otimes = \odot$ A is the category with objects all $[0,1]$-powered objects in the category $[0,1]$-GLat, and morphisms all those arrows in $[0,1]$-GLat that preserve powers by elements of $[0,1]$.

For the multiplication $\otimes = *$ A is the category with objects all $[0,1]$-powered objects in the category $[0,1]$-$\mathsf{GLat}_\ominus$, and morphisms all those arrows in $[0,1]$-$\mathsf{GLat}_\ominus$ that preserve powers by elements of $[0,1]$.

Remark 3.2.37. The category A over Set is a $\aleph_1$-ary quasivariety and, moreover, a full subcategory of a finitary variety. Therefore the isomorphisms in A are precisely the bijective morphisms.

Proposition 3.2.38. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Let $m \colon A \to CX$ be an injective morphism in A so that the cone $(m(a) \colon X \to [0,1]^{\mathrm{op}})_{a \in A}$ is point-separating and initial with respect to the forgetful functor into Set. Then m is an isomorphism in A if and only if A is Cauchy-complete.*

Proof. Clearly, if m is an isomorphism, then A is Cauchy-complete since CX is so. The reverse implication is clear for $\otimes = \odot$ by Lemmas 3.2.28 and 3.2.30. Consider now $\otimes = *$ multiplication. Let $\psi \in CX$. Put $\psi' = \frac{1}{2} * \psi + \frac{1}{2}$, then ψ' is monotone and continuous. By Lemmas 3.2.28 and 3.2.31, $\psi' \in \mathrm{im}(m)$ and therefore also $\psi = \hom(\frac{1}{2}, \psi' \ominus \frac{1}{2})$ belongs to $\mathrm{im}(m)$. □

Unfortunately we do not know if the $[0,1]$-category $[0,1]$, as an object of A, is a cogenerator. Therefore, we need to restrict the category A to the largest full subcategory of A where $[0,1]$ has this property. We say that an object A of A ***has enough characters*** whenever the cone $(\varphi\colon A \to [0,1])_\varphi$ of all morphisms into $[0,1]$ separates the points of A.

Theorem 3.2.39. *Let A be an object in A. Then $A \simeq CX$ in A for some separated ordered compact space X if and only if A is Cauchy-complete and has enough characters.*

Proof. If $A \simeq CX$ in A, then clearly A is Cauchy-complete and has enough characters. Assume now that A has these properties. Then, by [Lambek and Rattray, 1979, Proposition 2.4], $X = \hom(A, [0,1])$ is a separated ordered compact space with the initial structure relative to all evaluation maps $\mathrm{ev}_a\colon X \to [0,1]^{\mathrm{op}}$ $(a \in A)$. The map $m\colon A \to CX$, $a \mapsto ev_a$ is injective since A has enough characters and satisfies the hypothesis of Proposition 3.2.38, hence m is an isomorphism. □

Finally, Theorem 3.2.39 allows us to describe the image of the fully faithful functors of Theorem 3.2.16 and Corollary 3.2.19. We end this section presenting duality results for the categories $\mathsf{SepOrdComp}_{\mathbb{V}}$ and $\mathsf{SepOrdComp}$ where the objects on the dual side should be thought of as "metric distributive lattices". To do so, we consider now the following categories.

- $\mathsf{A}_{[0,1],\mathrm{cc}}$ denotes the full subcategory of A defined by the Cauchy-complete objects having enough characters.
- $\mathsf{B}_{[0,1],\mathrm{cc}}$ denotes the category with the same objects as $\mathsf{A}_{[0,1],\mathrm{cc}}$, and the morphisms of $\mathsf{B}_{[0,1],\mathrm{cc}}$ are the finitely cocontinuous $[0,1]$-functors which laxly preserve the multiplication.

Theorem 3.2.40. *For $\otimes = *$ the multiplication or $\otimes = \odot$ the Łukasiewicz tensor,*

$$\mathsf{SepOrdComp}_{\mathbb{V}} \simeq \mathsf{B}^{\mathrm{op}}_{[0,1],\mathrm{cc}} \qquad \textit{and} \qquad \mathsf{SepOrdComp} \simeq \mathsf{A}^{\mathrm{op}}_{[0,1],\mathrm{cc}}.$$

Proof. Follows from Theorem 3.2.39 and Remark 3.2.24. □

Chapter 4

Vietoris coalgebras

In this chapter we study properties of categories of coalgebras whose underlying functor is Vietoris polynomial, intuitively, a topological analogue of a Kripke polynomial functor; we call such coalgebras ***Vietoris coalgebras***. Part of this work has been published in [Hofmann et al., 2018a] [Hofmann et al., 2018b]. For an application in computer science, in the context of *hybrid programs*, see [Neves, 2018].

In Section 4.1 we prove that the category $\mathsf{CoAlg}(V)^{\mathrm{op}}$ of coalgebras for the Vietoris functor on $\mathsf{SepOrdComp}$ and certain full subcategories are $\aleph_1$-ary quasivarieties. Actually, the crucial step is to give a concrete presentation of the algebra structure of $\mathsf{SepOrdComp}^{\mathrm{op}}$ as an $\aleph_1$-ary quasivariety, and this is already partly proven in Section 3.2.2. The reader can find the pertinent definitions and results about quasivarieties in [Adámek and Rosický, 1994].

The results of Section 4.1 imply in particular that the category $\mathsf{CoAlg}(V)$ is complete. In Section 4.2.2, we deepen our understanding about limits in categories of Vietoris coalgebras by studying the whole class of Vietoris polynomial functors defined on Top. To prove the existence of limits, we essentially resort to Theorems 2.5.11 and 2.5.26. It turns out that a great deal of this section is devoted to the study of preservation of codirected limits by the compact and the lower Vietoris functors. In particular, we show that the categories of (suitably defined) Vietoris coalgebras over BooSp, $\mathsf{CompHaus}$, Priest, $\mathsf{SepOrdComp}$ and Haus are complete. Finally, we conclude this thesis by observing that categories of Vietoris coalgebras over Top have equalisers, (certain) codirected limits and, under some conditions, a terminal object.

4.1 The quasivariety $\mathsf{CoAlg}(V)^{\mathrm{op}}$

In Section 3.2.2 we saw that the category $\mathsf{SepOrdComp}^{\mathrm{op}}$ is embedded into an $\aleph_1$-ary quasivariety A of $[0,1]$-enriched categories that depend on the choice of the tensor. In this section we will only consider categories enriched in the quantale determined by the Łukasiewicz tensor in $[0,1]$.

We recall that the objects of A are the separated finitely cocomplete $[0,1]$-categories with a monoid structure that admit $[0,1]$-powers; the morphisms are the finitely cocontinuous $[0,1]$-functors preserving the monoid structure and the $[0,1]$-powers.

Theorem 4.1.1. *The functor*

$$C\colon \mathsf{SepOrdComp}^{\mathrm{op}} \longrightarrow \mathsf{A}$$

sending $f\colon X \to Y$ to $Cf\colon CY \to CX$, $\psi \mapsto \psi \cdot f$ is fully faithful, here the algebraic structure on

$$CX = \{f\colon X \to [0,1]^{\mathrm{op}} \mid f \text{ is monotone and continuous}\}$$

is defined pointwise.

In Theorem 3.2.40 we identified the image of the functor C, however, in way that does not allow us to immediately conclude that $\mathsf{SepOrdComp}^{\mathrm{op}}$ is an $\aleph_1$-ary quasivariety because we resorted to the notion of Cauchy completeness. Nevertheless, the following proposition is one important consequence of Theorem 3.2.38.

Proposition 4.1.2. *The unit interval $[0,1]$ is $\aleph_1$-copresentable in* $\mathsf{SepOrdComp}$.

Proof. This can be shown with the same argument as in [Gabriel and Ulmer, 1971, 6.5.(c)]. Firstly, by Theorem 3.2.39, $\hom(-,[0,1])$ sends every $\aleph_1$-codirected limit to a jointly surjective cocone. Secondly, using Theorem 2.5.3, this cocone is a colimit since $[0,1]$ is $\aleph_1$-copresentable in $\mathsf{CompHaus}$. □

Now, instead of working with Cauchy completeness we wish to add an operation to the algebraic theory of A such that, if M is closed under this operation in CX, then M is closed with respect to the topology of the $[0,1]$-category CX. As we observed in Remark 3.2.29, this topology coincides with the usual topology induced by the "sup-metric" on CX.

For compact Hausdorff spaces, the same problem is solved in Isbell [1982] using the operation

$$[0,1]^{\mathbb{N}} \longrightarrow [0,1],\ (u_n)_{n\in\mathbb{N}} \longmapsto \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u_n$$

on $[0,1]$. Given a compact Hausdorff space X, Isbell considers the set CX of all continuous functions $X \to [-1,1]$. He observes that every subset $M \subseteq CX$ closed under the operation above (defined now in $[-1,1]$), truncated addition and subtraction, is topologically closed. To see why, let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in M with limit $\varphi = \lim_{n\to\infty}\varphi_n$, we may assume that $\|\varphi_{n+1} - \varphi_n\| \le \frac{1}{2^{n+1}}$, for all $n \in \mathbb{N}$. Then

$$\varphi = \varphi_0 + \frac{1}{2}(2(\varphi_1 - \varphi_0)) + \cdots \in M.$$

However, this argument cannot be transported directly into the ordered setting since the difference $\varphi_1 - \varphi_0$ of two monotone maps $\varphi_0, \varphi_1 \colon X \to [0,1]$ is not necessarily monotone. To circumvent this problem, in the sequel we look for a monotone and continuous function $[0,1]^{\mathbb{N}} \to [0,1]$ that calculates the limit of "sufficiently many" sequences.

Lemma 4.1.3. *Let $M \subseteq CX$ be a subalgebra in A and $\psi \in CX$ with $\psi \in \overline{M}$. Then there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in M converging to ψ so that*

1. *$(\psi_n)_{n\in\mathbb{N}}$ is increasing, and*
2. *for all $n \in \mathbb{N}$ and all $x \in X$: $\psi_{n+1}(x) - \psi_n(x) \leq \frac{1}{2^n}$.*

Proof. We can find $(\psi_n)_{n\in\mathbb{N}}$ so that, for all $n \in \mathbb{N}$, $|\psi_n(x) - \psi(x)| \leq \frac{1}{n+1}$. Then the sequence $(\psi_n \ominus \frac{1}{n+1})_{n\in\mathbb{N}}$ converges to ψ too; moreover, since $M \subseteq CX$ is a subalgebra, also $\psi_n \ominus \frac{1}{n+1} \in M$, for all $n \in \mathbb{N}$. Therefore we can assume that we have a sequence $(\psi_n)_{n\in\mathbb{N}}$ in M with $(\psi_n)_{n\in\mathbb{N}} \to \psi$ and $\psi_n \leq \psi$, for all $n \in \mathbb{N}$. Then the sequence $(\psi_0 \vee \dots \vee \psi_n)_{n\in\mathbb{N}}$ has all its members in M, is increasing and converges to ψ. Finally, there is a subsequence of this sequence which satisfies the second condition above. □

Lemma 4.1.4. *Let*

$$\mathcal{C} = \{(u_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}} \mid (u_n)_{n\in\mathbb{N}} \text{ is increasing and } u_{n+1} - u_n \leq \frac{1}{2^n}, \text{ for all } n \in \mathbb{N}\}.$$

Then every sequence in $\mathcal{C}$ is Cauchy and $\lim \colon \mathcal{C} \to [0,1]$ is monotone and continuous.

Proof. Clearly, every element of $\mathcal{C}$ is a Cauchy sequence and the function $\lim \colon \mathcal{C} \to [0,1]$ is monotone. To see that lim is also continuous, let $(u_n)_{n\in\mathbb{N}} \in \mathcal{C}$ with and $\varepsilon > 0$. Put $u = \lim_{n\to\infty} u_n$. Choose $N \in \mathbb{N}$ so that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$ and $u - u_N < \frac{\varepsilon}{2}$. Then

$$U = \{(v_n)_{n\in\mathbb{N}} \in \mathcal{C} \mid |u - v_N| < \frac{\varepsilon}{2}\}$$

is an open neighbourhood of $(u_n)_{n\in\mathbb{N}}$. For every $(v_n)_{n\in\mathbb{N}} \in U$ with $v = \lim_{n\to\infty} v_n$,

$$|v - u| \leq |v - v_N| + |v_N - u| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;$$

which proves that $\lim \colon \mathcal{C} \to [0,1]$ is continuous. □

Motivated by the two lemmas above, we are looking for a monotone continuous map $[0,1]^{\mathbb{N}} \to [0,1]$ which sends every sequence in $\mathcal{C}$ to its limit. Such a map can be obtained by combining $\lim \colon \mathcal{C} \to [0,1]$ with a monotone continuous retraction of the inclusion map $\mathcal{C} \hookrightarrow [0,1]^{\mathbb{N}}$. The following result is straightforward to prove.

Lemma 4.1.5. *The map $\mu \colon [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}} \mapsto (u_0 \vee \dots \vee u_n)_{n\in\mathbb{N}}$ is monotone and continuous.*

Clearly, the map μ sends a sequence to an increasing sequence, and $\mu((u_n)_{n\in\mathbb{N}}) = (u_n)_{n\in\mathbb{N}}$ for every increasing sequence $(u_n)_{n\in\mathbb{N}}$.

Lemma 4.1.6. *The map $\gamma\colon [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ sending a sequence $(u_n)_{n\in\mathbb{N}}$ to the sequence $(v_n)_{n\in\mathbb{N}}$ defined recursively by*

$$v_0 = u_0 \qquad \text{and} \qquad v_{n+1} = \min\left(u_{n+1}, v_n + \frac{1}{2^n}\right)$$

is monotone and continuous. Furthermore, γ sends an increasing sequence to an increasing sequence.

Proof. It is easy to see that γ is monotone. To verify continuity, we consider $\mathbb{N}$ as a discrete topological space, this way $[0,1]^{\mathbb{N}}$ is an exponential in Top. We show that γ corresponds via the exponential law to a (necessarily continuous) map $f\colon \mathbb{N} \to [0,1]^{\left([0,1]^{\mathbb{N}}\right)}$. The recursion data above translate to the conditions

$$f(0) = \pi_0 \qquad \text{and} \qquad f(n+1)((u_m)_{m\in\mathbb{N}}) = \min\left(u_{n+1}, f(n)((u_m)_{m\in\mathbb{N}}) + \frac{1}{2^n}\right),$$

that is, f is defined by the recursion data $\pi_0 \in [0,1]^{\left([0,1]^{\mathbb{N}}\right)}$ and

$$[0,1]^{\left([0,1]^{\mathbb{N}}\right)} \longrightarrow [0,1]^{\left([0,1]^{\mathbb{N}}\right)},\ \varphi \longmapsto \min\left(\pi_{n+1}, \varphi + \frac{1}{2^n}\right).$$

Note that with $\varphi\colon [0,1]^{\mathbb{N}} \to [0,1]$ also $\min\left(\pi_{n+1}, \varphi + \frac{1}{2^n}\right) : [0,1]^{\mathbb{N}} \to [0,1]$ is continuous. Finally, if $(u_n)_{n\in\mathbb{N}}$ is increasing, then so is $(v_n)_{n\in\mathbb{N}}$. □

Therefore, the map $\gamma\cdot\mu\colon [0,1]^{\mathbb{N}} \to \mathcal{C}$ is a retraction for the inclusion map $\mathcal{C} \to [0,1]^{\mathbb{N}}$ in $\mathsf{SepOrdComp}$. Now we can add a new operation to algebraic theory of A.

Definition 4.1.7. Let $\overline{\mathsf{A}}$ be the $\aleph_1$-ary quasivariety obtained by adding one $\aleph_1$-ary operation symbol to the theory of A (see Remark 2.2.15). Then $[0,1]$ becomes an object of $\overline{\mathsf{A}}$ by interpreting this operation symbol by

$$\delta = \lim\cdot\gamma\cdot\mu\colon [0,1]^{\mathbb{N}} \to [0,1].$$

The (accordingly modified) functor $C\colon \mathsf{SepOrdComp} \to \overline{\mathsf{A}}$ is fully faithful (see Remark 3.2.24); moreover, by Proposition 4.1.2, C sends $\aleph_1$-codirected limits to $\aleph_1$-directed colimits in $\overline{\mathsf{A}}$.

Definition 4.1.8. Let $\overline{\mathsf{A}}_0$ be the subcategory of $\overline{\mathsf{A}}$ defined by those objects A that have enough characters; that is, where the cone of all morphisms from A to $[0,1]$ is point-separating.

The category $\overline{\mathsf{A}}_0$ is a regular epireflective full subcategory of $\overline{\mathsf{A}}$ and consequently also a quasivariety. Moreover:

Theorem 4.1.9. *The embedding* $C\colon \mathsf{SepOrdComp}^{\mathrm{op}} \to \overline{\mathsf{A}}$ *corestricts to an equivalence functor* $C\colon \mathsf{SepOrdComp}^{\mathrm{op}} \to \overline{\mathsf{A}}_0$*. Thus,* $\overline{\mathsf{A}}_0$ *is closed in* $\overline{\mathsf{A}}$ *under* $\aleph_1$*-directed colimits and therefore also an* $\aleph_1$*-ary quasivariety.*

From [Adámek and Rosický, 1994, Remark 3.32] it follows immediately.

Corollary 4.1.10. *The category* $\overline{\mathsf{A}}_0$ *is closed in* $\overline{\mathsf{A}}$ *under* $\aleph_1$*-directed colimits and therefore is also an* $\aleph_1$*-ary quasivariety.*

With the present discussion in mind, it is straightforward to verify that the category $\mathrm{CoAlg}(V)^{\mathrm{op}}$ of coalgebras for the Vietoris functor on $\mathsf{SepOrdComp}$ is an $\aleph_1$-ary quasivariety.

Let $\overline{\mathsf{B}}$ denote the category with the same objects as $\overline{A}$ and morphisms those maps $\varphi\colon A \to A'$ that preserve finite suprema and the action $- \odot u$, for all $u \in [0,1]$, and satisfy

$$\varphi(x \circledcirc y) \le \varphi(x) \circledcirc \varphi(y),$$

for all $x, y \in A$.

Theorem 4.1.11. *The functor* $C\colon \mathsf{SepOrdComp}^{\mathrm{op}} \to \overline{\mathsf{A}}$ *extends to a fully faithful functor* $C\colon \mathsf{SepOrdComp}_{\mathbb{V}} \to \overline{\mathsf{B}}$ *making the diagram*

$$\begin{array}{ccc} \mathsf{SepOrdComp}_{\mathbb{V}}^{\mathrm{op}} & \xrightarrow{C} & \overline{\mathsf{B}} \\ \uparrow & & \uparrow \\ \mathsf{SepOrdComp}^{\mathrm{op}} & \xrightarrow[C]{} & \overline{\mathsf{A}}_0 \end{array}$$

commutative, where the vertical arrows denote the canonical inclusion functors.

Proof. Follows from Theorem 3.2.23 and Remark 3.2.24. □

Clearly, a coalgebra structure $X \to VX$ for V can be also interpreted as an endomorphism $X \mathrel{-\!\!\!\circ\!\!\!\to} X$ in the Kleisli category $\mathsf{SepOrdComp}_{\mathbb{V}}$. Therefore the category $\mathrm{CoAlg}(V)$ is dually equivalent to the category with objects all pairs (A, a) consisting of an $\overline{\mathsf{A}}_0$ object A and a $\overline{\mathsf{B}}$-morphism $a\colon A \to A$, and a morphism between such pairs (A, a) and (A', a') is an $\overline{\mathsf{A}}_0$-morphism $A \to A'$ commuting in the obvious sense with a and a'.

Theorem 4.1.12. *The category* $\mathrm{CoAlg}(V)$ *of coalgebras and homomorphisms for the Vietoris functor* $V\colon \mathsf{SepOrdComp} \to \mathsf{SepOrdComp}$ *is dually equivalent to an* $\aleph_1$*-ary quasivariety.*

Proof. Just consider the algebraic theory of $\overline{\mathsf{A}}_0$ augmented by one unary operation symbol and by those equations which express that the corresponding operation is a $\overline{\mathsf{B}}$-morphism. □

In particular, $\mathrm{CoAlg}(V)$ is complete and the forgetful functor $\mathrm{CoAlg}(V) \to \mathsf{SepOrdComp}$ preserves $\aleph_1$-codirected limits. In fact, in the next sections we will see that slightly more is true.

We finish this section by exploring some further consequences of our approach for certain full subcategories of $\mathrm{CoAlg}(V)$. We are guided by familiar concepts, namely reflexive and transitive relations; but our arguments apply to other concepts as well, such as idempotent relations, for example.

Still thinking of a coalgebra structure $\alpha\colon X \to VX$ as an endomorphism $\alpha\colon X \nrightarrow X$ in $\mathsf{SepOrdComp}_{\mathbb{V}}$, we say that α is ***reflexive*** whenever $1_X \leq \alpha$ in $\mathsf{SepOrdComp}_{\mathbb{V}}$, and α is called ***transitive*** whenever $\alpha \circ \alpha \leq \alpha$ in $\mathsf{SepOrdComp}_{\mathbb{V}}$; with the local order in $\mathsf{SepOrdComp}_{\mathbb{V}}$ being inclusion.

Proposition 4.1.13. *The full subcategory of* $\mathrm{CoAlg}(V)$ *defined by all reflexive (or transitive or reflexive and transitive) coalgebras is dually equivalent to an* $\aleph_1$*-ary quasivariety. Moreover, this subcategory is coreflective in* $\mathrm{CoAlg}(V)$ *and closed under* $\aleph_1$*-directed limits.*

Proof. Clearly, the functor $C\colon \mathsf{SepOrdComp}_{\mathbb{V}} \to \overline{\mathsf{B}}$ preserves the local order of morphisms defined pointwise; by Proposition 2.3.9 it also reflects it. Therefore, considering the corresponding $\overline{\mathsf{B}}$-morphism $a\colon A \to A$, the inequalities expressing reflexivity and transitivity can be formulated as equations in A. Then the assertion follows from [Adámek and Rosický, 1994, Theorem 1.66]. □

For a class $\mathcal{M}$ of monomorphisms in $\mathrm{CoAlg}(V)$, a coalgebra X for V is called ***coorthogonal*** whenever, for all $m\colon A \to B$ in $\mathcal{M}$ and all homomorphisms $f\colon X \to B$ there exists a (necessarily unique) homomorphism $g\colon X \to A$ with $m \cdot g = f$ (see [Adámek and Rosický, 1994, Definition 1.32] for the dual notion). We write $\mathcal{M}^{\top}$ for the full subcategory of $\mathrm{CoAlg}(V)$ defined by those coalgebras which are coorthogonal to $\mathcal{M}$. From the dual of [Adámek and Rosický, 1994, Theorem 1.39] we obtain:

Proposition 4.1.14. *For every set* $\mathcal{M}$ *of monomorphisms in* $\mathrm{CoAlg}(V)$*, the inclusion functor* $\mathcal{M}^{\top} \hookrightarrow \mathrm{CoAlg}(V)$ *has a right adjoint. Moreover, if* λ *denotes a regular cardinal larger or equal to* $\aleph_1$ *so that, for every arrow* $m \in \mathcal{M}$*, the domain and codomain of* m *is* λ*-copresentable, then* $\mathcal{M}^{\top} \hookrightarrow \mathrm{CoAlg}(V)$ *is closed under* λ*-codirected limits.*

Another way of specifying full subcategories of $\mathrm{CoAlg}(V)$ uses coequations (see [Adámek, 2005, Definition 4.18]). For the Vietoris functor, the latter is a particular case of coorthogonality, and therefore we obtain the following result.

Corollary 4.1.15. *For every set of coequations in* $\mathrm{CoAlg}(V)$*, the full subcategory of* $\mathrm{CoAlg}(V)$ *defined by these coequations is coreflective.*

4.2 Limits in coalgebras

We start by introducing the notion of polynomial functor at a generic level; the set-based formulation and some applications can be found in Bonsangue et al. [2009]; Jacobs [2012].

Definition 4.2.1. Let C be distributive category. A ***polynomial functor*** on C is an element of the smallest class of endofunctors on C that contains the identity functor, all constant functors, and is closed under products and sums of functors. Here, for functors $F, G\colon \mathsf{C} \to \mathsf{C}$, the product of F and G, and the sum of F and G are, respectively, the composites

$$\mathsf{C} \xrightarrow{\langle F,G\rangle} \mathsf{C}\times\mathsf{C} \xrightarrow{\times} \mathsf{C}, \quad \text{and} \quad \mathsf{C} \xrightarrow{\langle F,G\rangle} \mathsf{C}\times\mathsf{C} \xrightarrow{+} \mathsf{C}.$$

Remark 4.2.2. Note that limits of a certain type that are preserved by the functors $F, G\colon \mathsf{C} \to \mathsf{C}$, are also preserved by the functor $F \times G\colon \mathsf{C} \to \mathsf{C}$.

Remark 4.2.3. A reader from computer science will quickly realise that a polynomial functor is recursively defined from the grammar below

$$F ::= F + F \mid F \times F \mid A \mid \mathrm{Id}$$

where A corresponds to an object of C.

In the next section we abstract slightly that polynomial functors in Top are liftings of polynomial functors in Set and show that categories of coalgebras of polynomial functors defined in *similar* categories admit the same type of limits. Then, in Section 4.2.2, we study limits in categories of coalgebras whose underlying functor is a topological analogue of the set theoretical notion of Kripke polynomial functor.

Definition 4.2.4. Let C be a distributive subcategory of Top such that the lower Vietoris functor (see Section 2.4) restricts to C. We call a functor ***lower Vietoris polynomial*** on Cif it belongs to the smallest class of endofunctors on C that contains the identity functor, all constant functors, the lower Vietoris functor and is closed under products and sums of functors.

Similarly, if we consider the compact Vietoris functor (see Section 2.4) instead of the lower one, then we speak of a ***compact Vietoris polynomial*** functor.

A ***Vietoris coalgebra*** is a coalgebra whose underlying functor is lower or compact Vietoris polynomial.

Our study of limits in categories of Vietoris coalgebras will consist essentially in determining conditions that guarantee completeness or, at least, that a terminal coalgebra exists. The following theorems summarise our basic strategic in each case; for more details see Section 2.5.

It is well-known that the category Top is regularly wellpowered, (co)complete and has an (Epi, RegMono)-factorisation structure (for example, see Adámek et al. [1990]). Therefore,

Corollary 4.2.5. *If a functor $F\colon \mathsf{Top} \to \mathsf{Top}$ preserves regular monomorphisms and codirected limits then the category of coalgebras of F is complete.*

In the sequel we will see that, in our case of study, the preservation of regular monomorphisms is not a hard problem. In fact, the theme of this section is to study the preservation of codirected limits by Vietoris functors in suitable subcategories of Top.

4.2.1 Polynomial functors

The case of polynomial functors in Top is straightforward. To apply Corollary 4.2.5 first we need to show that polynomial functors preserve regular monomorphisms and codirected limits. We can do it at once by proving that polynomial functors in Top preserve connected limits.

Proposition 4.2.6. *The functor* $(+)\colon \mathsf{Top}\times\mathsf{Top}\to\mathsf{Top}$ *preserves connected limits.*

Proof. It is well-known that the functor $(+)\colon \mathsf{Set}\times\mathsf{Set}\to\mathsf{Set}$ preserves connected limits , and it is simple to see that $(+)\colon \mathsf{Top}\times\mathsf{Top}\to\mathsf{Top}$ preserves initial cones. Therefore, the claim follows from Theorem 2.3.18. □

Corollary 4.2.7. *If functors* $F, G\colon \mathsf{Top}\to\mathsf{Top}$ *preserve connected limits, then the functor* $F+G\colon \mathsf{Top}\to\mathsf{Top}$ *preserves connected limits as well.*

Theorem 4.2.8. *Every polynomial functor* $F\colon \mathsf{Top}\to\mathsf{Top}$ *preserves connected limits.*

Proof. Clearly the identity functor $\mathrm{Id}\colon \mathsf{Top}\to\mathsf{Top}$ preserves all limits, and the constant functor $A\colon \mathsf{Top}\to\mathsf{Top}$ trivially preserves connected limits. The claim now follows from Remark 4.2.2 and Corollary 4.2.7. □

Proposition 4.2.9. *Every polynomial functor* $F\colon \mathsf{Top}\to\mathsf{Top}$ *preserves regular monomorphisms.*

Proof. A regular monomorphism is a limit of a connected diagram: a pair of parallel morphisms. □

Therefore,

Theorem 4.2.10. *If* $F\colon \mathsf{Top}\to\mathsf{Top}$ *is a polynomial functor, the category* $\mathsf{CoAlg}(F)$ *is complete.*

Motivated by the proximity between the cases of polynomial functors in Top and Set, we will be looking now for an indirect way of reasoning about limits in categories of coalgebras. The general idea is that starting with categories A and B linked by a functor that lifts "sufficiently many" cones in B, then for a functor $\overline{F}\colon \mathsf{A}\to\mathsf{A}$ for which there is a functor $F\colon \mathsf{B}\to\mathsf{B}$ that makes the diagram

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{\overline{F}} & \mathsf{A} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathsf{B} & \xrightarrow[F]{} & \mathsf{B} \end{array}$$

commute, we should be able to reason about limits in $\mathsf{CoAlg}(\overline{F})$ using our knowledge of limits in $\mathsf{CoAlg}(F)$. We begin by introducing the functor $\overline{U}\colon \mathsf{CoAlg}(\overline{F}) \to \mathsf{CoAlg}(F)$.

Proposition 4.2.11. *Let $U\colon \mathsf{A} \to \mathsf{B}$, $\overline{F}\colon \mathsf{A} \to \mathsf{A}$ and $F\colon \mathsf{B} \to \mathsf{B}$ be functors such that $U\overline{F} = FU$. The assignments*

$$\overline{U}(X, c) = (UX, Uc), \qquad \overline{U}f = Uf$$

define a functor $\overline{U}\colon \mathsf{CoAlg}(F) \to \mathsf{CoAlg}(\overline{F})$ that makes the diagram below commute.

$$\begin{array}{ccc} \mathsf{CoAlg}(\overline{F}) & \longrightarrow & \mathsf{A} \\ {\scriptstyle \overline{U}}\big\downarrow & & \big\downarrow{\scriptstyle U} \\ \mathsf{CoAlg}(F) & \longrightarrow & \mathsf{B} \end{array}$$

Conceptually, we want to start with a diagram in $\mathsf{CoAlg}(\overline{F})$, take its image under $\overline{U}$, calculate the corresponding limit in $\mathsf{CoAlg}(F)$ and then lift it to a limit in $\mathsf{CoAlg}(\overline{F})$. Technically, this creates two problems: we need to guarantee that the functor $U\colon \mathsf{A} \to \mathsf{B}$ lifts U-structured cones coming from limits in $\mathsf{CoAlg}(F)$ and that we can equip these lifts with an appropriate coalgebra structure. Of course, we can solve the first problem by forcing U to lift every U-structured cone, however, this disregards any useful information that we might know about limits in $\mathsf{CoAlg}(F)$. For example, we might be interested in limits that F preserves, or in limits that happen to be monocones in B. Therefore, we consider the next definition.

Definition 4.2.12. Let $U\colon \mathsf{A} \to \mathsf{B}$ be a functor and $\mathcal{M}$ a class of cones in B. The functor U is said to be ***$\mathcal{M}$-topological*** if every U-structured cone in $\mathcal{M}$ admits a U-initial lift. We denote by $U^{-1}\mathcal{M}$ the class of cones in A that are U-initial lifts of U-structured cones in $\mathcal{M}$.

The next theorem affirms that we can solve our second problem if $\overline{F}$ sends certain U-initial cones to U-initial cones.

Theorem 4.2.13. *Let $\mathcal{M}$ be a class of cones in B and $\mathcal{M}_{\mathsf{B}}$ the class of cones in $\mathsf{CoAlg}(F)$ whose underlying cone in B belongs to $\mathcal{M}$. If U is $\mathcal{M}$-topological and $\overline{F}$ sends U-initial cones in $U^{-1}\mathcal{M}$ to U-initial cones, then $\overline{U}$ is $\mathcal{M}_{\mathsf{B}}$-topological.*

Proof. Let $(f_i\colon (B, b) \to \overline{U}(A_i, a_i))_{i\in\mathsf{I}}$ be a cone in $\mathsf{CoAlg}(F)$ whose underlying cone is in $\mathcal{M}$. Then, by assumption, the cone $(f_i\colon B \to UA_i))_{i\in\mathsf{I}}$ admits a U-initial lift

$$(\overline{f}_i\colon A \to A_i)_{i\in\mathsf{I}}$$

to a cone in A. Thus, the cone $(\overline{F}\,\overline{f}_i\colon \overline{F}A \to \overline{F}A_i)_{i\in\mathsf{I}}$ is also U-initial, as $\overline{F}\colon \mathsf{A} \to \mathsf{A}$ preserves

U-initial cones in $U^{-1}\mathcal{M}$. Moreover, note that for every $i \in \mathsf{I}$ the following equations hold

$$U\left(A \overset{\overline{f}_i}{\to} A_i \overset{a_i}{\to} \overline{F}A_i\right) = B \overset{f_i}{\to} UA_i \overset{Ua_i}{\to} FUA_i$$
$$U\left(\overline{F}A \overset{\overline{F}\,\overline{f}_i}{\to} \overline{F}A_i\right) = FB \overset{Ff_i}{\to} FUA_i,$$

and that we have the factorisation

$$\begin{array}{ccc} B & \xrightarrow{f_i} & UA_i \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle Ua_i} \\ FB & \xrightarrow[Ff_i]{} & FUA_i. \end{array}$$

Since $(\overline{F}\,\overline{f}_i \colon \overline{F}A \to \overline{F}A_i)_{i\in\mathsf{I}}$ is U-initial, there is a coalgebra structure $a\colon A \to \overline{F}A$ such that $Ua = b$ and that the diagram below commutes.

$$\begin{array}{ccc} A & \xrightarrow{\overline{f}_i} & A_i \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a_i} \\ \overline{F}A & \xrightarrow[\overline{F}\,\overline{f}_i]{} & \overline{F}A_i \end{array}$$

Therefore, we obtain a cone $(\overline{f}_i \colon (A,a) \to (A_i,a_i))_{i\in\mathsf{I}}$ in $\mathsf{CoAlg}(\overline{F})$ whose image under $\overline{U}$ is $(f_i\colon (B,b) \to \overline{U}(A_i,a_i))_{i\in\mathsf{I}}$. To show that this cone is $\overline{U}$-initial, let $(g_i\colon (Z,z) \to (A,a_i))_{i\in\mathsf{I}}$ be a cone in $\mathsf{CoAlg}(\overline{F})$ such that its image under $\overline{U}$ factorises in $\mathsf{CoAlg}(F)$ as depicted below.

$$\begin{array}{ccc} \overline{U}(Z,z) & & \\ {\scriptstyle h}\downarrow & \searrow^{\overline{U}g_i} & \\ \overline{U}(A,a) & \xrightarrow[\overline{U}\,\overline{f}_i]{} & \overline{U}(A_i,a_i) \end{array}$$

Then, because $(\overline{f}_i\colon A \to A_i)_{i\in I}$ is U-initial, there is a unique morphism $\overline{h}\colon Z \to A$ in A such that $U\overline{h} = h$ and $g_i = \overline{f}_i \cdot \overline{h}$, for every $i \in \mathsf{I}$. This morphism is actually a morphism of type $(Z,z) \to (A,a)$ in $\mathsf{CoAlg}(\overline{F})$. To see why, observe that the follow diagram commutes

$$\begin{array}{ccccc} UZ & \xrightarrow{h} & B & \xrightarrow{f_i} & UA_i \\ {\scriptstyle Uz}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle Ua_i} \\ FUZ & \xrightarrow[Fh]{} & FB & \xrightarrow[Ff_i]{} & FUA_i \end{array},$$

and corresponds to the image under U of the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\overline{h}} & A & \xrightarrow{\overline{f}_i} & A_i \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_i} \\ \overline{F}Z & \xrightarrow[\overline{F}\,\overline{h}]{} & \overline{F}A & \xrightarrow[\overline{F}\,\overline{f}_i]{} & \overline{F}A_i \end{array} .$$

Since $(\overline{F}\,\overline{f}_i\colon \overline{F}A \to \overline{F}A_i)_{i\in\mathsf{I}}$ is U-initial, we conclude that the square on the left in the diagram above commutes, which means that $\overline{h}\colon (Z,z) \to (A,a)$ is a morphism in $\mathsf{CoAlg}(\overline{F})$. □

Under the conditions of the result above, we obtain

Corollary 4.2.14. *If the functor* $U\colon \mathsf{A} \to \mathsf{B}$ *is topological, then the functor* $\overline{U}\colon \mathsf{CoAlg}(\overline{F}) \to \mathsf{CoAlg}(F)$ *is topological as well.*

But even if functor $U\colon \mathsf{A} \to \mathsf{B}$ is not topological, we can show that some limits exist, assuming that U lifts "sufficiently many" cones. For example:

Corollary 4.2.15. *Take* $\mathcal{M}$ *as the class of limits in* B *of shape* I*. If* F *preserves limits of shape* I*, then* $\mathsf{CoAlg}(\overline{F})$ *has limits of shape* I*.*

The forgetful functor $\mathsf{Top} \to \mathsf{Set}$ is topological (see Adámek [2005]) and it is straightforward to show that every polynomial functor over Top preserves initial cones. This way, we obtain another proof for the fact that every category of coalgebras of a polynomial functor over Top is complete.

4.2.2 Vietoris polynomial functors

In Section 4.2.1 we studied limits in categories of polynomial coalgebras in Top, essentially by analysing the preservation of connected limits. The next examples reveal that the same strategy does not work for Vietoris polynomial functors.

Examples 4.2.16. 1. Consider $\mathsf{I} = \mathbb{N}$ with the natural order, and the functor $D\colon \mathbb{N} \to \mathsf{Set}$ that sends $n \leq m$ to the inclusion map $\{0,\dots n\} \hookrightarrow \{0,\dots,m\}$. Clearly, the set of natural numbers $\mathbb{N}$ is a colimit of this directed diagram. Then, the composite $\mathsf{Set}(-,\mathbb{N})\cdot D^{\mathrm{op}}\colon \mathbb{N}^{\mathrm{op}} \to \mathsf{Set}$ yields a codirected diagram with limit $\mathsf{Set}(\mathbb{N},\mathbb{N})$, the limit projections $p_n\colon \mathsf{Set}(\mathbb{N},\mathbb{N}) \to \mathsf{Set}(D(n),\mathbb{N})$ being given by restriction. We obtain a codirected limit in Top by equipping all sets with the indiscrete topology. The compact Vietoris functor does not send this limit to a monocone since $(Vp_n)_{n\in\mathbb{N}}$ cannot distinguish between the sets $\mathsf{Set}(\mathbb{N},\mathbb{N})$ and

$$\{f\colon \mathbb{N} \to \mathbb{N} \mid \{n \in \mathbb{N} \mid f(n) \neq 0\} \text{ is finite}\}.$$

2. This example is based on the "empty inverse limit" of Waterhouse [1972]. Take I as the set of all finite subsets of $\mathbb{R}$, with order being containment $\supseteq$. Consider the codirected diagram $D\colon \mathsf{I} \to \mathsf{Top}$ that sends every $F \in \mathsf{I}$ to the discrete space of all injective functions $D(F)$, and every map $G \supseteq F$ to the function $D(G \supseteq F)$ given by restriction. Note that each connecting map $D(G \supseteq F)$ is surjective. The limit of this diagram is necessarily empty otherwise its elements would define injective functions $\mathbb{R} \to \mathbb{N}$. The lower Vietoris functor sends the limit cone for D to a monocone but not to a limit cone since the limit of $\mathsf{V}D$ has at least two elements: $(\varnothing)_{F\in\mathsf{I}}$ and $(D(F))_{F\in\mathsf{I}}$. Using the indiscrete topology instead of the discrete, we can show that the lower Vietoris functor does not preserve codirected limits of diagrams of compact spaces and closed maps.

3. In the example above its is possible to use other topologies to show that the lower and the compact Vietoris functor do not preserve certain codirected limits. For example, consider $\mathbb{N}$ equipped with the topology

$$\{\uparrow n \mid n \in \mathbb{N}\} \cup \{\varnothing\};$$

Note that $\mathbb{N}$ is T_0 and every non-empty collection of open subsets of $\mathbb{N}$ has a largest element with respect to inclusion $\subseteq$. The latter implies that, for every finite set F, every subset of $\mathbb{N}^F$ is compact. To see why, let $C \subseteq \mathbb{N}^F$ and assume that C is covered by subbasic open subsets of $\mathbb{N}^F$:

$$C \subseteq \bigcup_{\lambda\in\Lambda} \pi_{i_\lambda}^{-1}[\uparrow n_\lambda].$$

Observe that the set $K = \{i_\lambda \mid \lambda \in \Lambda\} \subseteq F$ is finite. For every $i \in K$, let $k_i = \min\{n_\lambda \mid \lambda \in \Lambda, i_\lambda = i\}$. Then

$$C \subseteq \bigcup_{i\in K} \pi_i^{-1}[\uparrow k_i].$$

Therefore, by Alexander's Subbase Theorem (see [Kelley, 1975]), we conclude that C is compact.

With I defined as in the previous example, we consider now $D(F)$ as a subspace of $\mathbb{N}^F$. Then, for every $G \supseteq F$, the map $D(G \supseteq F)\colon D(G) \to D(F)$ is continuous. Thus, this construction defines a codirected diagram $D\colon \mathsf{I} \to \mathsf{Top}$ with empty limit where each $D(F)$ is T_0, compact, and locally compact; neither the lower nor the compact Vietoris functor preserve it, as we can see by following the same argument of the previous example.

Despite the examples above, in the sequel we will see that the Vietoris functors are well-behaved with respect to initial codirected monocones and regular monomorphisms.

Lemma 4.2.17. *Let X be a topological space and $\mathcal{B}$ a base for the topology of X.*

1. *The set $\{B^\Diamond \mid B \in \mathcal{B}\}$ is a subbase for the lower Vietoris space $\mathsf{V}X$ (see Section 2.4).*

2. *If $\mathcal{B}$ is closed under finite unions, then the set $\{B^\Diamond \mid B \in \mathcal{B}\} \cup \{B^\Box \mid B \in \mathcal{B}\}$ is a subbase for the compact Vietoris space $\mathsf{V}X$ (se Section 2.4).*

Proof. Let $\mathcal{S}$ be a set of open subsets of X. First note that, for both the lower and the compact Vietoris space,

$$\left(\bigcup \mathcal{S}\right)^\Diamond = \bigcup \{S^\Diamond \mid S \in \mathcal{S}\} .$$

This proves the first statement. To see that the second one is also true, observe that

$$\left(\bigcup \mathcal{S}\right)^\Box = \bigcup \left\{ \left(\bigcup \mathcal{F}\right)^\Box \mid \mathcal{F} \subseteq \mathcal{S} \text{ finite} \right\}$$

since we only consider compact subsets of X. □

Lemma 4.2.18. *Both the compact and the lower Vietoris functor $\mathsf{V}\colon \mathsf{Top} \to \mathsf{Top}$ preserve initial codirected cones.*

Proof. Let $(f_i\colon X \to X_i)_{i \in \mathsf{I}}$ be an initial codirected cone in Top. Then the set

$$\left\{ f_i^{-1}(U) \mid i \in I, U \subseteq X_i \text{ open} \right\}$$

is a base for the topology of X (Remark 2.3.17). Moreover, the base is closed under finite unions. Therefore, by the lemma above, the proof follows from the equations

$$((f_i)^{-1}(U))^\Box = (\mathsf{V}f_i)^{-1}(U^\Box) \qquad ((f_i)^{-1}(U))^\Diamond = (\mathsf{V}f_i)^{-1}\left(U^\Diamond\right),$$

for all $i \in \mathsf{I}$ and $U \subseteq X_i$ open, which are straightforward to show. □

Theorem 4.2.19. *The lower Vietoris functor preserves initial codirected monocones. The compact Vietoris functor preserves initial codirected monocones of Hausdorff spaces.*

Proof. First note that for a topological space X the lower Vietoris space $\mathsf{V}X$ is T_0, and if X is Hausdorff the compact Vietoris space $\mathsf{V}X$ is Hausdorff as well (see Michael [1951]) . Then recall that a initial cone in Top whose domain is T_0 (or T_2) is necessarily mono and apply Lemma 4.2.18. □

Remark 4.2.20. The assumption about codirectedness is essential: in general, neither the compact nor the lower Vietoris functor $\mathsf{V}\colon \mathsf{Top} \to \mathsf{Top}$ preserve monocones. Take, for instance, a compact Hausdorff space X with at least two elements. Then $\Delta = \{(x,x) \mid x \in X\}$ is a closed subset of $X \times X$, and Δ is different from $X \times X$. Therefore, with $\pi_1\colon X \times X \to X$ and $\pi_2\colon X \times X \to X$ denoting the projection maps,

$$\mathsf{V}\pi_1(\Delta) = \mathsf{V}\pi_1(X \times X) = X = \mathsf{V}\pi_2(\Delta) = \mathsf{V}\pi_2(X \times X);$$

which shows that the cone $(\mathsf{V}\pi_1\colon \mathsf{V}(X \times X) \to \mathsf{V}X, \mathsf{V}\pi_2\colon \mathsf{V}(X \times X) \to \mathsf{V}X)$ is not mono.

Together with Proposition 4.2.9 it follows:

Corollary 4.2.21. *Every compact Vietoris polynomial functor and every lower Vietoris polynomial functor defined on* Top *preserves regular monomorphisms.*

Proof. We already saw that all polynomial functors preserve regular monomorphisms in Proposition 4.2.9, and that the lower Vietoris functor preserves them as well in Theorem 4.2.19. Moreover, we saw that the compact Vietoris functor preserves initial codirected cones in Lemma 4.2.18, and it is straightforward to show that it preserves monomorphisms. □

From Theorem 4.2.19 and Corollary 2.5.22 we obtain the following results.

Corollary 4.2.22. *For every lower Vietoris polynomial functor* $F\colon \mathsf{Top} \to \mathsf{Top}$ *the category* $\mathsf{CoAlg}(F)$ *has codirected limits. For every compact Vietoris polynomial functor* $F\colon \mathsf{Top} \to \mathsf{Top}$ *the category* $\mathsf{CoAlg}(F)$ *has codirected limits of Hausdorff spaces.*

Proof. Let I be a codirected set and $\mathcal{M}$ the class of all initial monocones of shape I in Top. It is easy to see that polynomial functors preserve initial cones, and, by Theorem 4.2.19, the lower Vietoris functor preserves initial codirected monocones. Moreover, the category Top is $(Epi, InitialMono)$-structured for cones (for instance, see [Adámek et al., 1990, Examples 10.2(4) and 15.3(6)]) and it is clearly $\mathcal{M}$-wellpowered. Thus, the assertion follows from Theorem 2.5.22. The case of the compact Vietoris functor is analogous. □

Corollary 4.2.23. *For every Vietoris polynomial functor* $F\colon \mathsf{Top} \to \mathsf{Top}$*, the category of coalgebras* $\mathsf{CoAlg}(F)$ *has equalisers.*

Proof. Direct consequence of Theorem 2.5.25 and Corollary 4.2.21. □

The previous discussion highlights that we cannot apply Theorem 2.5.26 to every Vietoris polynomial functor on Top because, in general, Vietoris functors do not preserve codirected limits. We can fix this by focusing on a suitable subcategory of topological spaces. Since we still want to reason about terminal coalgebras over Top, it is natural to require that the diagram

$$1 \longleftarrow F1 \longleftarrow FF1 \longleftarrow \dots$$

in Top can be formed in such subcategory, and that the corresponding inclusion functor into Top behaves in a way that allows us to "import or export" properties about limits from or to Top. The next elementary propositions describe two useful situations.

Proposition 4.2.24. *Let* $\overline{F}\colon \mathsf{A} \to \mathsf{A}$*,* $F\colon \mathsf{B} \to \mathsf{B}$ *and* $U\colon \mathsf{A} \to \mathsf{B}$ *be functors such that* $U\overline{F} = FU$*. If* $\overline{F}$ *preserves a limit* L *and* U *preserves the limit* $\overline{F}(L)$*, then* F *preserves the limit* $U(L)$*.*

Proposition 4.2.25. *Let $\overline{F}\colon \mathsf{A} \to \mathsf{A}$, $F\colon \mathsf{B} \to \mathsf{B}$ and $U\colon \mathsf{A} \to \mathsf{B}$ be functors such that $U\overline{F} = FU$ and U preserves and reflects limits of shape I. If F preserves limits of shape I then $\overline{F}$ preserves limits of shape I.*

Proof. Consider a limit L of shape I. Since F and U preserve limits, $FU(L) = U\overline{F}(L)$ is a limit. Therefore, $F(L)$ is a limit because U reflects limits of shape I. □

In the sequel we will see that the category StablyComp of stably compact spaces and spectral maps serves our intents. As mentioned in Section 2.3 the inclusion functor StablyComp $\to$ Top is monadic (see Simmons [1982]), which in particular implies that it creates limits and reflects isomorphisms. Additionally, stably compact spaces can be described in terms of ordered topological structures. In fact, the category StablyComp is isomorphic to the category SepOrdComp of separated ordered compact spaces as described in Section 2.3. We recall from Proposition 2.4.3 that the counterpart of the lower Vietoris functor on SepOrdComp, that we also denote by V, sends a separated ordered compact space X to the space VX of all upper-closed subsets of X, with order containment $\supseteq$, and compact topology generated by the sets

$$\begin{aligned} &\{A \in VX \mid A \cap U \neq \varnothing\} \quad (U \subseteq X \text{ lower-open}),\\ &\{A \in VX \mid A \cap K = \varnothing\} \quad (K \subseteq X \text{ lower-closed}). \end{aligned} \tag{4.2.i}$$

Given a map $f\colon X \to Y$ in SepOrdComp, the functor V returns the map Vf that sends a upper-closed subset $A \subseteq X$ to the up-closure $\uparrow f[A]$ of $f[A]$. Thus, as a side effect, we can study preservation of limits by the compact Vietoris functor on CompHaus by studying preservation of limits by the lower Vietoris functor on StablyComp. The reason is that the compact Vietoris functor on CompHaus is the composite

$$\mathsf{CompHaus} \xrightarrow{\text{discrete}} \mathsf{SepOrdComp} \xrightarrow{\mathsf{V}} \mathsf{SepOrdComp} \xrightarrow{\text{forgetful}} \mathsf{CompHaus};$$

where, being right adjoints, the "discrete" and "forgetful" functors preserve limits

Therefore, we turn now to the specific problem of showing that the lower Vietoris functor $V\colon$ StablyComp $\to$ StablyComp preserves codirected limits.

A codirected limit of a diagram $D\colon \mathsf{I} \to$ StablyComp is given by the subspace

$$\left\{(x_i)_{i\in\mathsf{I}} \in \prod_{i\in\mathsf{I}} D(i) \mid \forall j \to i \in \mathsf{I}, D(j \to i)(x_j) = x_i\right\} \tag{4.2.ii}$$

of the product space $\prod_{i\in\mathsf{I}} D(i)$ together with the restrictions of the projection maps. And, for every limit cone $(p_i\colon L_D \to D(i))_{i\in\mathsf{I}}$, the canonical comparison map from $h\colon \mathsf{V}L_D \to L_{\mathsf{V}D}$ is defined by

$$K \mapsto (\overline{p_i[K]})_{i\in\mathsf{I}}.$$

In Theorem 4.2.19 we saw that the lower Vietoris functor preserves codirected initial monocones. This implies immediately that the comparison map $h\colon \mathsf{V}L_D \to L_{\mathsf{V}D}$ is an embedding. Then, to show that $\mathsf{V}\colon \mathsf{StablyComp} \to \mathsf{StablyComp}$ preserves codirected limits, we are left with the task of proving that h is also surjective. To do so, it seems easier to consider stably compact spaces as separated ordered compact spaces.

For compact Hausdorff spaces, we could invert h by observing that an element of $L_{\mathsf{V}D}$ defines a codirected diagram of surjective maps in $\mathsf{CompHaus}$, and that by taking its limit we get an element of $\mathsf{V}L_D$. We can use the same idea for arbitrary separated ordered compact spaces, but then an element of $L_{\mathsf{V}D}$ defines a codirect diagram of "order dense" maps. This procedure still works because, as we will see next, the category $\mathsf{SepOrdComp}$ inherits the nice characterisation of codirected limits described in Theorem 2.5.3.

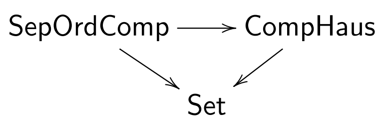
Proposition 4.2.26. *Let $\mathcal{A}$ be a codirected set of closed subsets of a separated ordered compact space X. Then, $\uparrow\bigcap_{A\in\mathcal{A}} A = \bigcap_{A\in\mathcal{A}} \uparrow A$.*

Proof. Clearly, $\uparrow\bigcap_{A\in\mathcal{A}} A \subseteq \bigcap_{A\in\mathcal{A}} \uparrow A$. To show that the reverse inequality holds, consider $z \in \bigcap_{A\in\mathcal{A}} \uparrow A$. Then, for every $A \in \mathcal{A}$, the set $\downarrow z \cap A$ is non-empty, and closed because $\{z\}$ is compact (see Proposition 2.3.7). Moreover, since $\mathcal{A}$ is codirected, the set $\{\uparrow z \cap A \mid A \in \mathcal{A}\}$ has the finite intersection property. Therefore, by compactness, it follows that $\downarrow z \cap \bigcap_{A\in\mathcal{A}} A \neq \varnothing$, which implies that $z \in \uparrow\bigcap_{A\in\mathcal{A}} A$. □

Proposition 4.2.27. *Let $D\colon \mathsf{I} \to \mathsf{SepOrdComp}$ be a codirected diagram, $(p_i\colon L_D \to D(i))_{i\in\mathsf{I}}$ a limit for D and $(L_{\mathsf{V}D} \to \mathsf{V}D(i))_{i\in\mathsf{I}}$ a limit for $\mathsf{V}D\colon \mathsf{I} \to \mathsf{SepOrdComp}$. Then the function $h\colon \mathsf{V}L_D \to L_{\mathsf{V}D}$ defined by $K \mapsto (\uparrow p_i[K])_{i\in\mathsf{I}}$ is surjective.*

Proof. Let $(K_i)_{i\in\mathsf{I}} \in L_{\mathsf{V}D}$. For every $i \in \mathsf{I}$, $K_i \subseteq D(i)$ is closed, hence, $K_i \in \mathsf{SepOrdComp}$. For every $i \in \mathsf{I}$ and $j \to i \in \mathsf{I}$, take $K(i)$ as K_i and $K(j \to i)$ as the continuous and monotone map of type $K_j \to K_i$ given by the restriction of $D(j \to i)$ to K_j. This way, by the description 4.2.ii , we obtain a codirected diagram $K\colon I \to \mathsf{SepOrdComp}$ such that for every $j \to i \in \mathsf{I}$, $\uparrow K(j \to i)[K(j)] = [K(i)]$.

Let $(p_i\colon L_K \to K(i))_{i\in\mathsf{I}}$ be a limit for K. By construction, $L_K \subseteq L_D$ is upper-closed. Thus, $L_K \in \mathsf{V}L_D$. We claim that $h(L_k) = (K_i)_{i\in\mathsf{I}}$. Let $i_0 \in I$. Since the following diagram of forgetful functors



commutes and the functor $\mathsf{SepOrdComp} \to \mathsf{CompHaus}$ preserves limits, from Theorem 2.5.3 we obtain

$$p_{i_0}[L_K] = \bigcap_{j\to i_0} K(j \to i_0)[K_j].$$

Therefore, by Proposition 4.2.26,

$$\uparrow p_{i_0}[L_K] = \uparrow \bigcap_{j \to i_0} K(j \to i_0)[K(j)] = \bigcap_{j \to i_0} \uparrow K(j \to i_0)[K(j)] = K_{i_0}.$$

□

As expected, we obtain

Theorem 4.2.28. *The lower Vietoris functor* $\mathsf{V}\colon \mathsf{StablyComp} \to \mathsf{StablyComp}$ *preserves codirected limits.*

Accordingly, for the category of compact Hausdorff spaces it follows

Corollary 4.2.29. *The compact Vietoris* $\mathsf{V}\colon \mathsf{CompHaus} \to \mathsf{CompHaus}$ *preserves codirected limits.*

The starting point of the idea behind the proof of Proposition 4.2.27 is that an element of $L_{\mathsf{V}D}$ defines a codirected diagram of compact Hausdorff spaces. Of course, we do not need our space to be compact to end up in such a situation, we just need it to be Hausdorff. Therefore, with the same idea, we have

Theorem 4.2.30 (Zenor [1970])**.** *The compact Vietoris functor* $\mathsf{V}\colon \mathsf{Haus} \to \mathsf{Haus}$ *preserves codirected limits.*

An extensive study of Vietoris coalgebras can be also found in Kupke et al. [2004] and [Bonsangue et al., 2007]. The former considers compact Vietoris polynomial functors on the category BooSp, and the latter coalgebras for the lower Vietoris functor on the category Spec. To relate the results of this section with Kupke et al. [2004] and [Bonsangue et al., 2007], in the sequel we will see that the lower Vietoris functor on Spec and the compact Vietoris functor on BooSp preserve codirected limits. In particular, this will allow us to show that the category of coalgebras of a compact Vietoris polynomial functor on the category BooSp is complete, which expands a result of Kupke et al. [2004] that guarantees the existence of a terminal coagebra. Again, we can study both cases simultaneously: as described at the end of Section 2.4, the compact Vietoris on BooSp is the composite of the functors

$$\mathsf{BooSp} \xrightarrow{\text{discrete}} \mathsf{Priest} \simeq \mathsf{Spec} \xrightarrow{\mathsf{V}} \mathsf{Spec} \simeq \mathsf{Priest} \xrightarrow{\text{forgetful}} \mathsf{BooSp};$$

where being right adjoints, the "discrete" and "forgetful" functors preserve limits.

Theorem 4.2.31. *The lower Vietoris functor* $V\colon \mathsf{Spec} \to \mathsf{Spec}$ *preserves codirected limits.*

Proof. The lower Vietoris polynomial functor on Spec is the restriction of the lower Vietoris functor on $\mathsf{StablyComp}$ (see Proposition 2.4.2). By Proposition 2.3.23, the functor $\mathsf{Spec} \to \mathsf{StablyComp}$ preserves and reflects limits, therefore, the assertion follows from Proposition 4.2.25. □

Accordingly, for the category BooSp we obtain

Corollary 4.2.32. *The compact Vietoris functor $V\colon$ BooSp $\to$ BooSp preserves codirected limits.*

Getting back to limits in categories of Vietoris coalgebras, to apply Theorem 2.5.26 first we need to show that Vietoris polynomial functors preserve codirected limits.

Proposition 4.2.33. *Every lower Vietoris polynomial functor on* StablyComp *or* Spec *preserves codirected limits. Similarly, every compact Vietoris polynomial functor on the categories* Haus*,* CompHaus *or* BooSp *preserves codirected limits.*

Proof. We have already seen in Corollaries 4.2.29, 4.2.32, and Theorems 4.2.28 4.2.31 4.2.30 that the lower and the compact Vietoris functors on the appropriate categories preserve codirected limits. Regarding polynomial functors, we can proceed as described in Theorem 4.2.8 for the category Top, since the inclusion functors into Top preserve finite coproducts (see Theorem 2.3.4 and Proposition 2.3.25). □

Therefore,

Theorem 4.2.34. *The category of coalgebras of a lower Vietoris polynomial functor on* StablyComp *or* Spec *is complete. Similarly, the category of coalgebras of a compact Vietoris polynomial functor on* Haus*,* CompHaus *or* BooSp *is complete.*

Proof. Being an epireflective subcategory of Top (for example, see [Adámek et al., 1990]), the category Haus is complete, cocomplete, and wellpowered. Moreover, it has a (Surjection, Embedding)-structure because it is a full subcategory of Top closed under subspace embeddings in Top. Thus, from Proposition 2.3.25 and Theorems 2.3.4, 2.3.14, we conclude that each category of the theorem satisfies the assumptions of Theorem 2.5.26. We have already seen that compact and lower Vietoris polynomial functors preserve subspace embeddings in Corollary 4.2.21 and that they preserve codirected limits in the corresponding categories in Proposition 4.2.33. Therefore, the assertion follows from Theorem 2.5.26. □

Regarding Vietoris polynomial functors on Top, we can improve slightly our results.

Proposition 4.2.35. *Every lower Vietoris polynomial functor on* Top *that restricts to the category* StablyComp *admits a terminal coalgebra. Similarly, every compact Vietoris polynomial functor on* Top *that restricts to* Haus *admits a terminal coalgebra.*

Proof. For a lower Vietoris polynomial functor $F\colon$ Top $\to$ Top satisfying the condition of the proposition, the diagram

$$1 \longleftarrow F1 \longleftarrow FF1 \longleftarrow \dots$$

in Top can be formed in the category StablyComp. Therefore, the assertion follows from Proposition 4.2.24 and Theorem 2.5.11 because the inclusion functor StablyComp $\to$ Top preserves limits. The claim for compact Vietoris polynomial functors follows in a similar way since the inclusion functor Haus $\to$ Top also preserves limits (see [Makkai and Paré, 1990]). □

Finally, recalling Corollaries 4.2.22 and 4.2.23 we conclude:

Theorem 4.2.36. *The category of coalgebras of a lower Vietoris polynomial on* Top *has equalisers and codirect limits; moreover, if the functor restricts to* StablyComp*, it has a terminal object.*

Theorem 4.2.37. *The category of coalgebras of a compact Vietoris polynomial on* Top *has equalisers and codirect limits of Hausdorff spaces; moreover, if the functor restricts to* Haus*, it has a terminal object.*

Bibliography

Adámek, J. (2005). Introduction to coalgebra. *Theory and Applications of Categories*, 14(8):157–199.

Adámek, J., Herrlich, H., and Strecker, G. E. (1990). *Abstract and concrete categories: The joy of cats*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York. Republished in: Reprints in Theory and Applications of Categories, No. 17 (2006) pp. 1–507.

Adámek, J. and Rosický, J. (1994). *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge.

Adámek, J., Rosický, J., and Vitale, E. (2010). *Algebraic Theories*, volume 184 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge.

Alsina, C., Frank, M. J., and Schweizer, B. (2006). *Associative functions. Triangular norms and copulas*. Hackensack, NJ: World Scientific.

Appelgate, H., Barr, M., Beck, J., Lawvere, F. W., Linton, F. E. J., Manes, E., Tierney, M., and Ulmer, F. (1969). *Seminar on Triples and Categorical Homology Theory*, volume 80 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg.

Banaschewski, B. (1983). On lattices of continuous functions. *Quaestiones Mathematicæ*, 6(1-3):1–12.

Barr, M. and Wells, C. (1985). *Toposes, triples and theories*. Springer-Verlag New York. Republished in: Reprints in Theory and Applications of Categories, No. 12, 2005, pp. 1 – 288.

Beck, J. M. (1967). *Triples, algebras and cohomology*. PhD thesis, Columbia University. Republished in: Reprints in Theory and Applications of Categories, No. 2 (2003) pp 1-59.

Bonsangue, M. M., Kurz, A., and Rewitzky, I. M. (2007). Coalgebraic representations of distributive lattices with operators. *Topology and its Applications*, 154(4):778–791.

Bonsangue, M. M., Rutten, J. J. M. M., and Silva, A. (2009). An algebra for kripke polynomial coalgebras. In *Proceedings of the 24th Annual IEEE Symposium on Logic in Computer Science, LICS 2009, 11-14 August 2009, Los Angeles, CA, USA*, pages 49–58. IEEE Computer Society.

Bourbaki, N. (1942). *Éléments de mathématique. 3. Pt. 1: Les structures fondamentales de l'analyse. Livre 3: Topologie générale.* Paris: Hermann & Cie.

Bourbaki, N. (1966). *General topology, part I.* Hermann, Paris and Addison-Wesley. Chapters 1–4.

Chen, L.-T. (2014). *On a purely categorical framework for coalgebraic modal logic.* PhD thesis, University of Birmingham.

Clark, D. M. and Davey, B. A. (1998). *Natural dualities for the working algebraist*, volume 57 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge.

Clementino, M. M. and Hofmann, D. (2003). Topological features of lax algebras. *Applied Categorical Structures*, 11(3):267–286.

Clementino, M. M. and Hofmann, D. (2009). Relative injectivity as cocompleteness for a class of distributors. *Theory and Applications of Categories*, 21(12):210–230.

Clementino, M. M. and Tholen, W. (1997). A characterization of the Vietoris topology. In *Proceedings of the 12th Summer Conference on General Topology and its Applications (North Bay, ON, 1997)*, volume 22, pages 71–95.

Clementino, M. M. and Tholen, W. (2003). Metric, topology and multicategory—a common approach. *Journal of Pure and Applied Algebra*, 179(1-2):13–47.

Dimov, G. D. and Tholen, W. (1989). A characterization of representable dualities. In Adámek, J. and MacLane, S., editors, *Categorical topology and its relation to analysis, algebra and combinatorics: Prague, Czechoslovakia, 22-27 August 1988*, pages 336–357. World Scientific.

Eilenberg, S. and Kelly, G. M. (1966). Closed categories. In Eilenberg, S., Harrison, D., Röhrl, H., and MacLane, S., editors, *Proceedings of the Conference on Categorical Algebra: La Jolla 1965*, pages 421–562. Springer Verlag.

Eilenberg, S. and Moore, J. C. (1965). Adjoint functors and triples. *Illinois Journal of Mathematics*, 9(3):381–398.

Engelking, R. (1989). *General topology*, volume 6 of *Sigma Series in Pure Mathematics.* Heldermann Verlag, Berlin, 2 edition. Translated from the Polish by the author.

Escardó, M. H. (1997). Injective spaces via the filter monad. In *Proceedings of the 12th Summer Conference on General Topology and its Applications (North Bay, ON, 1997)*, volume 22, pages 97–100.

Faucett, W. M. (1955). Compact semigroups irreducibly connected between two idempotents. *Proceedings of the American Mathematical Society*, 6(5):741–747.

Flagg, R. C. (1992). Completeness in continuity spaces. In Seely, R. A. G., editor, *Category Theory 1991: Proceedings of an International Summer Category Theory Meeting, held June 23-30, 1991*, volume 13 of *CMS Conference Proceedings*. American Mathematical Society.

Gabriel, P. and Ulmer, F. (1971). *Lokal präsentierbare Kategorien.* Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin.

Gelfand, I. (1941). Normierte Ringe. *Recueil Mathématique. Nouvelle Série*, 9(1):3–24.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W., and Scott, D. S. (1980). *A compendium of continuous lattices.* Springer-Verlag, Berlin.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W., and Scott, D. S. (2003). *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge.

Goubault-Larrecq, J. (2013). *Non-Hausdorff topology and domain theory*, volume 22. Cambridge University Press (CUP).

Halmos, P. R. (1956). Algebraic logic I. Monadic Boolean algebras. *Compositio Mathematica*, 12:217–249.

Hofmann, D. (1999). *Natürliche Dualitäten und das verallgemeinert Stone-Weierstraß Theorem.* PhD thesis, University of Bremen.

Hofmann, D., Neves, R., and Nora, P. (2018a). Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces. *Theory and Applications of Categories*, 33(12):276–295.

Hofmann, D., Neves, R., and Nora, P. (2018b). Limits in categories of Vietoris coalgebras. *Mathematical Structures in Computer Science (in press).*

Hofmann, D. and Nora, P. (2015). Dualities for modal algebras from the point of view of triples. *Algebra Universalis*, 73(3):297–320.

Hofmann, D. and Nora, P. (2018). Enriched Stone-type dualities. *Advances in Mathematics*, 330:307–360.

Hofmann, D. and Tholen, W. (2010). Lawvere completion and separation via closure. *Applied Categorical Structures*, 18(3):259–287.

Huber, P. J. (1961). *Homotopy theory in general categories*. PhD thesis.

Hughes, J. (2001). *A Study of Categories of Algebras and Coalgebras*. PhD thesis, Carnegie Mellon University.

Isbell, J. R. (1982). Generating the algebraic theory of $C(X)$. *Algebra Universalis*, 15(2):153–155.

Jacobs, B. (2012). Introduction to coalgebra. Towards mathematics of states and observations.

Johnstone, P. T. (1986). *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge. Reprint of the 1982 edition.

Jung, A. (2004). Stably compact spaces and the probabilistic powerspace construction. In Desharnais, J. and Panangaden, P., editors, *Proceedings of the Workshop on Domain Theoretic Methods for Probabilistic Processes*, volume 87 of *Electronic Notes in Theoretical Computer Science*, pages 5–20.

Kelley, J. L. (1975). *General topology*, volume 27 of *Graduate texts in mathematics*. Springer-Verlag, New York.

Kelly, G. M. (1982). *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge. Republished in: Reprints in Theory and Applications of Categories. No. 10 (2005), 1–136.

Kelly, G. M. and Lack, S. (2000). On the monadicity of categories with chosen colimits. *Theory and Applications of Categories*, 7(7):148–170.

Kelly, G. M. and Schmitt, V. (2005). Notes on enriched categories with colimits of some class. *Theory and Applications of Categories*, 14(17):399–423.

Kleisli, H. (1965). Every standard construction is induced by a pair of adjoint functors. *Proceedings of the American Mathematical Society*, 16(3):544–546.

Kock, A. (1971). On double dualization monads. *Mathematica Scandinavica*, 27(2):151–165.

Kupke, C., Kurz, A., and Venema, Y. (2004). Stone coalgebras. *Theoretical Computer Science*, 327(1-2):109–134.

Lambek, J. and Rattray, B. A. (1978). Functional completeness and Stone duality. In Rota, G.-C., editor, *Studies in foundations and combinatorics*, pages 1–9. Academic Press, New York.

Lambek, J. and Rattray, B. A. (1979). A general Stone–Gelfand duality. *Transactions of the American Mathematical Society*, 248(1):1–35.

Lawson, J. (2011). Stably compact spaces. *Mathematical Structures in Computer Science*, 21(1):125–169.

Lawvere, F. W. (1973). Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matemàtico e Fisico di Milano*, 43(1):135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

Linton, F. E. J. (1969). Coequalizers in categories of algebras. In Appelgate et al. [1969], pages 75–90.

MacLane, S. (1971). *Categories for the working mathematician.* Springer-Verlag, New York. Graduate Texts in Mathematics, Vol. 5.

Makkai, M. and Paré, R. (1990). *Accessible Categories: The Foundations of Categorical Model Theory.* American Mathematical Society.

Manes, E. G. (1969). A triple theoretic construction of compact algebras. In Appelgate et al. [1969], pages 91–118.

Michael, E. (1951). Topologies on spaces of subsets. *Transactions of the American Mathematical Society*, 71(1):152–182.

Mostert, P. S. and Shields, A. L. (1957). On the structure of semi-groups on a compact manifold with boundary. *Annals of Mathematics. Second Series*, 65(1):117–143.

Nachbin, L. (1950). *Topologia e Ordem.* University of Chicago Press.

Nachbin, L. (1965). *Topology and order.* Number 4 in Van Nostrand Mathematical Studies. D. Van Nostrand, Princeton, N.J.-Toronto, Ont.-London. Translated from the Portuguese by Lulu Bechtolsheim.

Negrepontis, J. W. (1971). Duality in analysis from the point of view of triples. *Journal of Algebra*, 19(2):228–253.

Neves, R. (2018). *Hybrid Programs.* PhD thesis, Minho Aveiro Porto – Doctoral Programme in Computer Science.

Porst, H.-E. and Tholen, W. (1991). Concrete dualities. In Herrlich, H. and Porst, H.-E., editors, *Category theory at work*, volume 18 of *Research and Exposition in Mathematics*, pages 111–136. Heldermann Verlag, Berlin. With Cartoons by Marcel Erné.

Priestley, H. A. (1970). Representation of distributive lattices by means of ordered stone spaces. *Bulletin of the London Mathematical Society*, 2(2):186–190.

Priestley, H. A. (1972). Ordered topological spaces and the representation of distributive lattices. *Proceedings of the London Mathematical Society. Third Series*, 24(3):507–530.

Pumplün, D. (1970). Eine Bemerkung über Monaden und adjungierte Funktoren. *Mathematische Annalen*, 185(4):329–337.

Pumplün, D. (1988). Eilenberg–Moore algebras revisited. *Seminarberichte, FB Mathematik u. Inf., Fernuniversität Hagen*, 29:97–144.

Raney, G. N. (1952). Completely distributive complete lattices. *Proceedings of the American Mathematical Society*, 3(5):677–680.

Rutten, J. (2000). Coalgebra, concurrency, and control. In *Discrete Event Systems*, pages 31–38. Springer Science & Business Media.

Schalk, A. (1993). *Algebras for Generalized Power Constructions.* PhD thesis, Technische Hochschule Darmstadt.

Shapiro, L. B. (1992). On function extension operators and normal functors. *Vestnik Moskovskogo Universiteta. Seriya I. Matematika, Mekhanika*, (1):35–42.

Simmons, H. (1982). A couple of triples. *Topology and its Applications*, 13(2):201–223.

Stone, M. H. (1936). The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society*, 40(1):37–111.

Stone, M. H. (1948a). The generalized Weierstrass approximation theorem. *Mathematics Magazine*, 21(4):167–184.

Stone, M. H. (1948b). The generalized Weierstrass approximation theorem. *Mathematics Magazine*, 21(5):237–254.

Stubbe, I. (2005). Categorical structures enriched in a quantaloid: categories, distributors and functors. *Theory and Applications of Categories*, 14(1):1–45.

Stubbe, I. (2006). Categorical structures enriched in a quantaloid: tensored and cotensored categories. *Theory and Applications of Categories*, 16(14):283–306.

Stubbe, I. (2007). $\mathcal{Q}$-modules are $\mathcal{Q}$–suplattices. *Theory and Applications of Categories*, 19(4):50–60.

Tholen, W. (1974). *Relative Bildzerlegungen und algebraische Kategorien.* PhD thesis, Westfälische Wilhelms-Universität Münster.

Tholen, W. (2009). Ordered topological structures. *Topology and its Applications*, 156(12):2148–2157.

Vietoris, L. (1922). Bereiche zweiter Ordnung. *Monatshefte für Mathematik und Physik*, 32(1):258–280.

Waterhouse, W. C. (1972). An empty inverse limit. *Proceedings of the American Mathematical Society*, 36(2):618.

Wood, R. J. (2004). Ordered Sets via Adjunction. In Pedicchio, M. C. and Tholen, W., editors, *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory*, volume 97 of *Encyclopedia of Mathematics and its Applications*, pages 5–47. Cambridge University Press, Cambridge.

Zenor, P. (1970). On the completeness of the space of compact subsets. *Proceedings of the American Mathematical Society*, 26(1):190–192.