On the construction of multi-valued concurrent dynamic logic

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Abstract. Dynamic logic is a powerful framework for reasoning about imperative programs. An extension with a concurrent operator [18] was introduced to formalise programs running in parallel. In other direction, other authors proposed a systematic method for generating multi-valued propositional dynamic logics to reason about weighted programs [14]. This paper presents the first step of combining these two frameworks to introduce uncertainty in concurrent computations. In the developed framework, a weight is assigned to each branch of the parellel execution, resulting in a (possible) asymmetric parallelism, inherent to fuzzy programming paradigm [21,2]. By adopting such an approach, a family of logics is obtained, called *multi-valued concurrent propositional dynamic logics* ($CGDL(\mathbf{A})$), parametric on an action lattice \mathbf{A} specifying a notion of "weight" assigned to program execution. Additionally, the validity of some axioms of CPDL is discussed in the new family of generated logics.

1 Introduction

Over time, the different variations of dynamic logics developed went hand-inhand with the very notion of its object, the *program*. This resulted in a diverse myriad of dynamic logics tailored to specific programming paradigms. Examples include probabilistic [11], concurrent [18], quantum [1] and continuous [19] computations, and combinations thereof. An example of another non-trivial paradigm is the fuzzy one [21,2], where the execution of a program differs from both classical and probabilistic scenarios: a conditional statement may act as a concurrent execution with a weight associated to each branch. The formalisation of such behaviour encompasses two non-trivial computational settings: concurrency and uncertainty. An extensive research can be found in the literature on diverse formalisms to reason about programs running in parallel [9,10] and to deal with uncertainty [11,5,20,4]. However, even when these two components are combined into a single framework [16], the uncertainty models probabilistic nondeterminism. Thus we are still missing a proper semantics to describe the behaviour of the fuzzy paradigm.

Recently, reference [14] initiated a research agenda on the systematic development of multi-valued propositional dynamic logics, parametric on an action

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lattice, which defines both the computational paradigm where programs live, and the truth space where assertions are evaluated. Following another research line, an extension to propositional dynamic logic (PDL) was introduced in reference [18], called *concurrent propositional dynamic logic (CPDL)*, to reason about concurrent computations. In the models presented for this logic, the programs are interpreted as *binary multirelations*, to describe a parallel execution from a state to a set of states.

Combining these research lines, this paper takes the first step on the development of a method to generate multi-valued concurrent propositional dynamic logics. As in [14], the logics are parametric on a generic action lattice, to model both the computational domain and a (possible graded) truth space where the assertions about programs are evaluated. First, the semantics of CPDL is adapted to model programs as weighted parallel executions, by introducing the concept of *fuzzy multirelations*. That means that a program, in the new logics, is interpreted as a relation between a state and a fuzzy set of states. The intuition is that the weights of the fuzzy set may describe an execution probability of each branch of the program, an asymmetric parallel flow or even the energy/costs associated to each branch. The second step of this paper consists on presenting the actual method of generating (parametric) multi-valued CPDL. The family of the resulting logics is called $CGDL(\mathbf{A})$.

This paper is organised as follows. Section 2 presents a brief background overview. Then, Section 3 starts to introduce fuzzy multirelations and defines some operations over them. Such algebra is the mathematical formalism in which programs are interpreted in the generated logics. The same section ends with the study of an axiomatisation for the generated logics. Finally, Section 4 concludes and enumerates topics for future work.

2 Preliminaries

2.1 Semantics for concurrency

The semantics of CPDL is based on the concept of *binary multirelation*. The relevant definition and some operators are recalled below.

Definition 1 (Binary multirelation [7]). Given a set X, a binary multirelation is a subset of the cartesian product $X \times P(X)$, i.e. a set of ordered pairs (a, A), where $a \in X$ and $A \subseteq X$. The following operations over multirelations are defined:

- $R \cup S$ as the union of R and S;
- the Peleg sequential composition

$$R \cdot S = \Big\{ (a, A) \mid \exists B.(a, B) \in R \land \exists f.(\forall b \in B.(b, f(b)) \in S) \land A = \bigcup f(B) \Big\};$$

- the parallel composition $R \cap S = \{(a, A \cup B) \mid (a, A) \in R \land (a, B) \in S\}.$

Note that the union of binary multirelations is just the set union. The sequential composition operator is rather more complex. A pair (a, A) belongs to the sequential composition of multirelations R and S if and only if a is related with some intermediate set of states B and every $b \in B$ must be related with some subset of A such that the union of all those subsets is A. Finally, an element $(a, A) \in R \cap S$ indicates a parallel execution of a program from a state ato a set of states in A, "combining" the arriving states of R and S into A. Note that such composition is dual to $R \cup S$, where (a, B) and (a, C) correspond to distinct executions. The first kind of choice in commonly called *demonic*, while the latter is known as *angelic*.

2.2 Concurrent propositional dynamic logic

Concurrent propositional dynamic logic (CPDL), as introduced in [18], is an extension of PDL with a parallel operator \cap added to the syntax of programs. The semantics interprets programs as binary multirelations $R \subseteq W \times P(W)$, where composed programs are interpreted according to the operators of Definition 1. Intuitively, an element (a, A) of a binary multirelation expresses that the a program executed from a state a ends in all states of A in parallel. Models of CPDL consist of tuples $(W, V, \llbracket - \rrbracket)$ where W is a set of states, V is a valuation function which attributes a subset of W to each atomic formula, and $\llbracket - \rrbracket$ attributes a subset of $W \times P(W)$ to each atomic program. For instance, the formula $\langle \pi \rangle \rho$ holds in a state w if and only if $\exists U \subseteq W$ s.t. $(s, U) \in \llbracket \pi \rrbracket$ and $U \in V(\rho)$. For more details about the semantics of CPDL see [18]. The axiom system of CPDL is that of PDL with the additional axiom $\langle \pi_1 \cap \pi_2 \rangle \rho \equiv \langle \pi_1 \rangle \rho \wedge \langle \pi_2 \rangle \rho$ and restricting $\langle \pi_0 \rangle (\rho \vee \rho') \equiv \langle \pi_0 \rangle \vee \langle \pi_0 \rangle \rho'$ to atomic programs.

2.3 Parametric construction of multi-valued dynamic logics

Thus subsection provides a short review of the dynamisation method introduced in [14]. Let us start by revisiting the following definition:

Definition 2 ([12]). An action lattice is a tuple $\mathbf{A} = (A, +, ;, 0, 1, *, \rightarrow, \cdot)$, that is a residuated lattice with order \leq induced by $+: a \leq b$ iff a + b = b, plus the axioms $1 + a + (a^*; a^*) \leq a^*$ and $(x \rightarrow x)^* = x \rightarrow x$.

An action lattice is called a \mathbb{I} -action lattice when the identity of the ; operator coincides with the greatest element of the residuated lattice, i.e. $1 = \top$. Moreover, an action lattice **A** is complete when every subset of **A** has both supremum and infimum. Since operators + and ; are associative, we can generalise them to n-ary operators and use the notation \sum and \prod to represent their iterated versions, respectively. The generation of dynamic logics illustrated in the Section 3 will be parametric on the class of complete action lattices, since completeness is required to ensure the existence of arbitrary suprema. The general construction of multi-valued dynamic logics is revisited below.

Signatures. Signatures of $\mathcal{GDL}(\mathbf{A})$ are pairs (Π , Prop) corresponding to the denotations of atomic programs and propositions, respectively.

Formulæ. The set of composed programs, denoted by $\operatorname{Prg}(\Pi)$, contains all expressions generated by $\pi \ni \pi_0 | \pi; \pi | \pi + \pi | \pi^*$ for $\pi_0 \in \Pi$. Given a signature $(\Pi, \operatorname{Prop})$, the $\mathcal{GDL}(\mathbf{A})$ -formulæ for $(\Pi, \operatorname{Prop})$ are the ones generated by the grammar $\rho \ni \top | \bot | p | \rho \lor \rho | \rho \land \rho | \rho \to \rho | \rho \leftrightarrow \rho | \langle \pi \rangle \rho | [\pi] \rho$ for $p \in \operatorname{Prop}$ and $\pi \in \operatorname{Prg}(\Pi)$.

Semantics. The space where the computations of $\mathcal{GDL}(\mathbf{A})$ are interpreted is given by the algebra $\mathbb{M}_n(\mathbf{A}) = (M_n(\mathbf{A}), +, ;, \mathbf{0}, \mathbf{1}, *)$ where $M_n(\mathbf{A})$ is the space of $(n \times n)$ -matrices over \mathbf{A} , the operators +, ; are the usual matrix sum and multiplication, respectively, $\mathbf{0}$, $\mathbf{1}$ are the zero matrix and the identity matrix, respectively, and * is the operator defined as in [3,13]. The matrix representation of a program expresses, for each pair of states s, s', the weight (e.g. probability, cost, uncertainty) of the program going from s to s'.

 $\mathcal{GDL}(\mathbf{A})$ -models for a signature (Prop, Π), denoted by $\mathrm{Mod}^{\mathcal{GDL}(\mathbf{A})}(\Pi, \mathrm{Prop})$, consists of tuples $\mathcal{A} = (W, V, (\mathcal{A}_{\pi})_{\pi \in \Pi})$ where W is a finite set (of states), $V : \mathrm{Prop} \times W \to A$ is a valuation function, and $\mathcal{A}_{\pi} \in M_n(\mathbf{A})$, with n standing for the cardinality of W.

The interpretation of a program $\pi \in Prg(\Pi)$ in a model $\mathcal{A} \in Mod^{\mathcal{GDL}(\mathbf{A})}(\Pi, Prop)$ is recursively defined, from the set of atomic programs $(\mathcal{A}_{\pi})_{\pi \in \Pi}$, as $\mathcal{A}_{\pi;\pi'} = \mathcal{A}_{\pi}$; $\mathcal{A}_{\pi'}, \mathcal{A}_{\pi+\pi'} = \mathcal{A}_{\pi} + \mathcal{A}_{\pi'}$ and $\mathcal{A}_{\pi^*} = \mathcal{A}_{\pi}^*$.

Satisfaction. The (graded) satisfaction relation, for a model $\mathcal{A} \in \operatorname{Mod}^{\mathcal{GDL}(\mathbf{A})}(\Pi, \operatorname{Prop})$, with **A** complete, consists of a function $\models : W \times \operatorname{Fm}^{\Gamma(\mathbf{A})}(\Pi, \operatorname{Prop}) \to A$ recursively defined as follows:

$$- (w \models \top) = \top$$
$$- (w \models \bot) = \bot$$
$$- (w \models \bot) = V(p, w), \text{ for any } p \in \text{Prop}$$
$$- (w \models \rho \land \rho') = (w \models \rho) \cdot (w \models \rho')$$
$$- (w \models \rho \lor \rho') = (w \models \rho) + (w \models \rho')$$
$$- (w \models \rho \to \rho') = (w \models \rho) \to (w \models \rho')$$
$$- (w \models \rho \leftrightarrow \rho') = (w \models \rho \to \rho'); (w \models \rho' \to \rho)$$
$$- (w \models \langle \pi \rangle \rho) = \sum_{w' \in W} (\mathcal{A}_{\pi}(w, w'); (w' \models \rho))$$
$$- (w \models [\pi]\rho) = \prod_{w' \in W} (\mathcal{A}_{\pi}(w, w') \to (w' \models \rho))$$

The (graded) satisfaction in a given state gives the degree of certainty of a formula in such state. For instance $M, w \models \langle \pi \rangle \rho$ gives the certainty that ρ is achieved from state w through the execution of π . It is relevant to note that $\mathcal{GDL}(\mathbf{A})$ is a generalisation of PDL, for each action lattice \mathbf{A} . In particular, by considering the Boolean lattice, the generated logic $\mathcal{GDL}(\mathbf{2})$ coincides with PDL.

Multi-valued concurrent dynamic logic 3

Before presenting the construction of the logic, we introduce the mathematical formalism to define the model where the programs will be interpreted.

Fuzzy binary multirelations 3.1

Definition 3 (Fuzzy set [22]). Given a set X and a complete residuated lattice **L**, a fuzzy subset of X is a function $\phi: X \to L$; $\phi(x)$ defines the membership degree of x in ϕ . The set of all fuzzy subsets of X is denote as L^X . The support of ϕ is a fuzzy subset ψ such that $\psi(x) > 0$. $\forall x \in X$.

Since an action lattice is an extension of a residuated lattice, the concept of fuzzy set can be defined as well for the former. Such is the case for all the remaining formalisms introduced in this paper.

Definition 4 (Fuzzy binary multirelation). Given a set X and a complete action lattice **A** over carrier A, a fuzzy binary multirelation R over X is a set $R \subseteq X \times A^X$. The following operations for fuzzy binary multirelations are defined:

- $R \cup S$ as the union of R and S;
- $-R \cdot S = \left\{ (a, \phi) \mid \phi(c) = \sum_{(a, \phi_a) \in R} \left(\prod_{(b, \phi_b) \in S} \phi_a(b); \phi_b(c) \right) \right\}$ $-R \cap S = \left\{ (a, \phi_R \cup \phi_S) \mid (a, \phi_R) \in R \text{ and } (a, \phi_S) \in S \right\}, \text{ where } \phi_R \cup \phi_S \text{ is the } b \in S \in S \right\}$ union of fuzzy sets ϕ_R and ϕ_s , as defined in [22];
- $R^* = \bigcup \{ R^n : n \ge 0 \}.$

We denote by M(X) the set of all fuzzy binary multirelations over X.

Note, particularly, how this definition generalises the concept of binary multirelations, particularly to the role of lattice A. This structure supports a set of truth values beyond the classical true and false, which are associated to the elements of the second component of R. By using such formalisation we are able to model a program as an execution with multiple "arrows" leaving a state to a set of states in parallel, with a (possible different) fuzziness degree associated with each "arrow".Note that if A is the Boolean lattice 2, any fuzzy multirelation $R \subseteq X \times \mathbf{2}^X$ is a binary multirelation. Since the goal is still to model programs as binary input-output relations, only the binary case is considered, and thus the remaining of this paper refers to fuzzy binary multirelations simply as fuzzy multirelations. Another aspect that is relevant for the formalisation of the logics is the restriction to fuzzy multirelations $R \subseteq X \times A^X$ where the fuzzy set ϕ in A^X is defined such that $\phi(x) > 0, \forall x \in X$. In other words, we take only the support of fuzzy sets for the fuzzy multirelations considered in this paper.

The operations for fuzzy multirelations are interpreted buying intuitions from the classic definition. One such case is the operator \cup , which corresponds to the classical set union. Regarding the sequential composition, the expression for ϕ computes the weight of an execution that starts from a state a, arrives at a set of intermediate states ϕ_a and ends in a set of states φ_b . The parallel composition considers the union of fuzzy sets for computing the external choice, which is just a generalisation of the set union used for CPDL.

3.2 Parametric construction of multi-valued concurrent dynamic logics

Each complete action lattice \mathbf{A} induces a multi-valued, concurrent propositional dynamic logic $\mathcal{CGDL}(\mathbf{A})$, with weighted computations interpreted over \mathbf{A} . Its signature, formulæ, semantics and satisfacton are presented below.

Signatures. Signatures of $CGDL(\mathbf{A})$ are pairs (Π , Prop) corresponding to the denotations of atomic programs and propositions, respectively.

Formulæ. The set of composed programs, denoted by $\operatorname{Prg}(\Pi)$, consists of all expressions generated by $\pi \ni \pi_0 | \pi; \pi | \pi \cap \pi | \pi + \pi | \pi^*$, for $\pi_0 \in \Pi$. Given a signature (Π , Prop), the $\mathcal{CGDL}(\mathbf{A})$ -formulæ for (Π , Prop), denoted by $\operatorname{Fm}^{\Gamma(\mathbf{A})}(\Pi, \operatorname{Prop})$, are the ones generated by the grammar $\rho \ni \top | \perp | p | \rho \lor \rho | \rho \land \rho | \rho \to \rho | \rho \leftrightarrow \rho | \langle \pi \rangle \rho$, for $p \in \operatorname{Prop}$ and $\pi \in \operatorname{Prg}(\Pi)$.

Semantics. The space where the programs are interpreted is given by the set of all fuzzy multirelations over a set of states W, denoted by M(W), and the operations over its elements, as in Definition 4.

 $\mathcal{CGDL}(\mathbf{A})$ -models for a signature $(\Pi, \operatorname{Prop})$ are tuples $M = (W, V, \llbracket - \rrbracket)$ where W is a set of states, V is a valuation function $V : \operatorname{Prop} \times W \to A$ and $\llbracket - \rrbracket$ attributes a fuzzy multirelation $R \subseteq W \times A^W$ to each atomic program.

The interpretation of a program $\pi \in Prg(\Pi)$ in a model M is recursively defined as:

 $[\![\pi;\pi']\!] = [\![\pi]\!] \cdot [\![\pi']\!], [\![\pi \cap \pi']\!] = [\![\pi]\!] \cap [\![\pi']\!], [\![\pi + \pi']\!] = [\![\pi]\!] \cup [\![\pi']\!] \text{ and } [\![\pi^*]\!] = [\![\pi]\!]^*.$

The satisfaction relation for a model $M = (W, V, \llbracket - \rrbracket)$ is given by the valuation function $\models_{\mathcal{CGDL}}: W \times \operatorname{Fm}^{\Gamma(\mathbf{A})}(\Pi, \operatorname{Prop}) \to A$ recursively defined as:

 $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \top) = \top$ $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \bot) = \bot$ $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} p) = V(p, w), \text{ for any } p \in \operatorname{Prop}$ $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho \land \rho') = (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) \cdot (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho')$ $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho \lor \rho') = (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) + (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho')$ $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho \to \rho') = (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) \to (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho')$ $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho \leftrightarrow \rho') = (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho \to \rho'); (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho' \to \rho)$ $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \langle \pi \rangle \rho) = \sum_{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket} \left(\prod_{u \in U} (\phi(u); (u \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho)) \right)$ $- (w \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \llbracket \pi \rbrack \rho) = \prod_{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket} \left(\prod_{u \in U} (\phi(u) \to (u \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho)) \right)$

where $U \subseteq W$. We say that ρ is *valid* when, for any model M, and for each state $w \in W$, $(w \models_{\mathcal{CGDL}} \rho) = \top$.

The satisfaction of formula $(w \models_{\mathcal{CGDL}} \langle \pi \rangle \rho)$ is given by the weight of some fuzzy set ϕ which is related with state w by some fuzzy multirelation, and that of ρ for every state of the domain of ϕ . Moreover, the satisfaction for the box operator follows [14], where every execution of the program must lead to a set of states all of which satisfy ρ . As mentioned in Section 2, the axiomatisation of CPDL was presented as being that of PDL, except for one that is restricted to atomic programs, plus an additional axiom for concurrency. Below we study such axiomatisation in the new models presented for $CGDL(\mathbf{A})$.

First, Lemma 1 provides some auxiliary properties used to prove next lemma.

Lemma 1. Let \mathcal{A} be a complete \mathbb{I} -action lattice. Then

(1.1) $(w \models_{\mathcal{CGDL}} \rho \to \rho') = \top iff (w \models_{\mathcal{CGDL}} \rho) \leq (w \models_{\mathcal{CGDL}} \rho')$ (1.2) $(w \models_{\mathcal{CGDL}} \rho \leftrightarrow \rho') = \top iff (w \models_{\mathcal{CGDL}} \rho) = (w \models_{\mathcal{CGDL}} \rho')$

Proof. Analogous to [14].

Lemma 2. Let \mathbf{A} be a complete \mathbb{I} -action lattice. The following are valid formulæ in any $CGDL(\mathbf{A})$:

(2.1) $\langle \pi_0 \rangle (\rho \lor \rho') \leftrightarrow \langle \pi_0 \rangle \rho \lor \langle \pi_0 \rangle \rho'$ (2.2) $\langle \pi \rangle (\rho \land \rho') \rightarrow \langle \pi \rangle \rho \land \langle \pi \rangle \rho'$ (2.3) $\langle \pi + \pi' \rangle \rho \leftrightarrow \langle \pi \rangle \rho \lor \langle \pi \rangle \rho$ (2.4) $\langle \pi \rangle \bot \leftrightarrow \bot$ (2.5) $\langle \pi \cap \pi' \rangle \rho \leftrightarrow \langle \pi \rangle \rho \land \langle \pi' \rangle \rho$ (2.6) $[\pi + \pi'] \rho \leftrightarrow [\pi] \rho \land [\pi'] \rho$ (2.7) $[\pi] (\rho \land \rho') \rightarrow [\pi] \rho \land [\pi] \rho'$

Proof. The proof uses the satisfaction function \models_{CGDL} and some axioms and properties of action lattices. The technical details are documented in Appendix A.

4 Conclusion

We took, in this paper, the first step in order to develop a rigorous and systematic formalism for the verification of weighted concurrent systems, motivated by the fuzzy case. The approach is based on the combination of some ideas from previous research [15,18,8] to characterise both the computational and logical settings on top of which a proper (axiomatic, denotational and operational) semantics for fuzzy programs will be developed, in future work.

There are numerous research lines that were left open and are worth to pursue in the near future. The most obvious is the study of a proper complete axiomatisation for the generated logics. In particular, the validity of the remaining axioms of CPDL, namely the ones involving operators ; and *, will be analysed in the new models. Another relevant path to be followed would be to study the relations between PDL, CPDL and their graded variants. In one direction, we propose to investigate whether CPDL can be obtained from $CGDL(\mathbf{A})$ by taking **2** as lattice. Other would be to study if there is a way to obtain multi-valued PDL as special case of $CGDL(\mathbf{A})$, such that there is a correspondence between the operations for fuzzy multirelations and operations on matrices. Additionally, relevant results about decidability and complexity of the logics are naturally in our agenda.

Although we base our definition of sequential composition for fuzzy multirelations in that of Peleg, there are other versions of the operator worth to be analysed. One corresponds to the definition introduced for giving semantics to Parikh's game logic [17]

$$R \cdot S = \Big\{ (a, A) \mid \exists B.(a, B) \in R \land \exists f.(\forall b \in B.(b, A) \in S) \Big\}$$

It is clearly stronger than Peleg's, since it requires that every intermediate state b must be related with the arriving set of states A. Another one, the Kleisli composition, was later studied in [6]. It is our goal to introduce proper generalisations of such operations, with possible applications in scenarios like a graded variant of game logics, as well as the development of axiomatic systems for each variation.

Finally, we propose to adapt the models of the generated logics in order to allow the introduction of assignments of variables to values in a given data domain. The goal is to develop (parametric) logics for the verification of programs written in a fuzzy imperative programming language, such as [21] or [2].

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A Appendix

The proofs here presented resort to some axioms of action lattices, enumerated below, and properties stated by Lemma 3.

$$a + (b + c) = (a + b) + c$$

$$a + b = b + a$$

$$a + 0 = 0 + a = a$$

$$a; (b + c) = (a; b) + (a; c)$$

$$a; 0 = 0; a = 0$$

Lemma 3. The following properties hold for any action lattice A:

$$a \le b \& c \le d \Rightarrow a + c \le b + d \tag{1}$$

$$a; (b \cdot c) \le (a; b) \cdot (a; c) \tag{2}$$

For I finite, we also have

$$\sum_{i \in I} (a_i \cdot b_i) \le \sum_{i \in I} a_i \cdot \sum_{i \in I} b_i \tag{3}$$

Proof of Lemma 2

(2.1):

$$\begin{split} w &\models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \langle \pi \rangle (\rho \lor \rho')) \\ &= \begin{cases} \text{ definition of }\models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rbrace \\ \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket}} \left(\prod_{u \in U} \left(\phi(u); (u \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho \lor \rho') \right) \right) \\ &= \begin{cases} \text{ definition of }\models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rbrace \\ \text{ definition of }\models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rbrace \\ \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket}} \left(\prod_{u \in W} \left((\phi(u); (u \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) + (u \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho') \right) \right) \\ &= \begin{cases} (1) \rbrace \\ \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket}} \left(\prod_{w' \in W} \left(\phi(w'); (w' \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho' \right) \right) \\ \\ &= \begin{cases} (1) \rbrace \\ \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket}} \left(\prod_{w' \in W} \left(\phi(w'); (w' \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) + (\psi \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) + (\psi \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) + (\psi \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) + (\psi \models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rho) \\ \\ &= \begin{cases} \text{ definition of }\models_{\mathcal{C}\mathcal{G}\mathcal{D}\mathcal{L}} \rangle \\ \\ &= \end{cases} \end{cases} \end{split}$$

Therefore, by Lemma 1, $\langle \pi \rangle (\rho \lor \rho') \leftrightarrow \langle \pi \rangle \rho \lor \langle \pi \rangle \rho$ is valid.

(2.2):

$$\begin{split} & \left(w \models_{\mathcal{CGDL}} \langle \pi \rangle (\rho \land \rho') \right) \\ &= \left\{ \begin{array}{l} \text{definition of } \models_{\mathcal{CGDL}} \right\} \\ & \sum_{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket} \left(\sum_{w' \in W} \left(\phi(w'); (w' \models_{\mathcal{CGDL}} \rho \land \rho') \right) \right) \\ &= \left\{ \begin{array}{l} \text{definition of } \models_{\mathcal{CGDL}} \right\} \\ & \sum_{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket} \left(\sum_{w' \in W} \left(\phi(w'); \\ (w' \models_{\mathcal{CGDL}} \rho) \cdot (w' \models_{\mathcal{CGDL}} \rho') \right) \right) \\ &\leq \left\{ \begin{array}{l} \text{by (2) and (1)} \right\} \\ & \sum_{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket} \left(\sum_{w' \in W} \left((\phi(w'); (w' \models_{\mathcal{CGDL}} \rho) \right) \right) \\ &= \left\{ \begin{array}{l} \text{definition of } \models_{\mathcal{CGDL}} \langle \pi \rangle \rho \rangle \cdot (w \models_{\mathcal{CGDL}} \langle \pi \rangle \rho') \\ & \left((w' \models_{\mathcal{CGDL}} \langle \pi \rangle \rho \wedge \langle \pi \rangle \rho' \right) \\ & \left(\phi(w'); (w' \models_{\mathcal{CGDL}} \rho') \right) \end{array} \right) \end{split}$$

Therefore, by Lemma 1, $\langle \pi \rangle (\rho \land \rho') \to \langle \pi \rangle \rho \land \langle \pi \rangle \rho'$ is valid.

$$(2.3)$$
:

$$\begin{array}{l} (w \models_{\mathcal{CGDL}} \langle \pi + \pi' \rangle \rho) \\ = & \left\{ \begin{array}{l} \operatorname{definition of } \models_{\mathcal{CGDL}} \right\} \\ & \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi + \pi' \rrbracket}} \left(\sum_{w' \in W} \left(\phi(w'); (w' \models_{\mathcal{CGDL}} \rho) \right) \right) \\ = & \left\{ \begin{array}{l} \operatorname{definition of } \llbracket \pi + \pi' \rrbracket \right\} \\ & \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket}} \left(\sum_{w' \in W} \left(\phi(w'); (w' \models_{\mathcal{CGDL}} \rho) \right) \right) \\ & = & \left\{ \begin{array}{l} \operatorname{definition of } \llbracket \pi + \pi' \rrbracket \right\} \\ & \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi' \rrbracket}} \left(\sum_{w' \in W} \left(\phi(w'); (w' \models_{\mathcal{CGDL}} \rho) \right) \right) \\ & = & \left\{ \begin{array}{l} \operatorname{definition of } \models_{\mathcal{CGDL}} \\ \operatorname{definition of } \models_{\mathcal{CGDL}} \\ (w \models \langle \pi \rangle \rho) + (w \models \langle \pi' \rangle \rho) \\ & = & \left\{ \begin{array}{l} \operatorname{definition of } \models_{\mathcal{CGDL}} \\ \operatorname{definition of } \models_{\mathcal{CGDL}} \\ (w \models \langle \pi \rangle \rho \lor \langle \pi' \rangle \rho) \end{array} \right. \end{array} \right.$$

Therefore, by Lemma 1, $\langle \pi + \pi' \rangle \rho \leftrightarrow \langle \pi \rangle \rho \vee \langle \pi' \rangle \rho$ is valid.

$$(2.4)$$
:

$$\begin{array}{l} \left(w \models_{\mathcal{CGDL}} \langle \pi \rangle \bot\right) \\ = & \left\{ \begin{array}{l} \operatorname{definition of} \models_{\mathcal{CGDL}} \right\} \\ \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket}} \left(\sum_{w' \in W} \left(\phi(w'); (w' \models_{\mathcal{CGDL}} \bot) \right) \right) \\ = & \left\{ \begin{array}{l} \operatorname{definition of satisfaction} \right\} \\ \sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi \rrbracket}} \left(\sum_{w' \in W} \left(\phi(w'); \bot \right) \right) \end{array} \right) \\ = & \left\{ \begin{array}{l} \operatorname{by} (1) \right\} \\ = & \left\{ \begin{array}{l} \operatorname{by} (1) \right\} \\ \operatorname{by} (1) \right\} \\ \bot \end{array} \right. \end{array}$$

Therefore, by Lemma 1, $\langle \pi \rangle \bot \leftrightarrow \bot$ is valid.

(2.5):

$$w \models_{\mathcal{CGDL}} \langle \pi \cap \pi' \rangle \rho$$

$$= \begin{cases} \text{definition of } \models_{\mathcal{CGDL}} \end{cases}$$

$$\sum_{\substack{\phi \mid (w,\phi) \in \llbracket \pi \cap \pi' \rrbracket}} \left(\sum_{w' \in W} (\phi(w'); w' \models_{\mathcal{CGDL}} \rho) \right) \\ = & \{ \text{definition of } \llbracket \pi \cap \pi' \rrbracket \} \end{cases}$$

$$\sum_{\substack{\phi_1 \mid (w,\phi_1) \in \llbracket \pi \rrbracket}} \left(\sum_{w' \in W} (\phi_1(w'); w' \models_{\mathcal{CGDL}} \rho) \right) \\ \cdot & \sum_{\substack{\phi_2 \mid (w,\phi_2) \in \llbracket \pi' \rrbracket}} \left(\sum_{w' \in W} (\phi_2(w'); w' \models_{\mathcal{CGDL}} \rho) \right) \\ = & \{ \text{definition of } \models_{\mathcal{CGDL}} \rbrace \\ (w \models_{\mathcal{CGDL}} \langle \pi \rangle \rho) \cdot (w \models_{\mathcal{CGDL}} \langle \pi' \rangle \rho) \\ = & \{ \text{definition of } \models_{\mathcal{CGDL}} \rbrace \\ (w \models_{\mathcal{CGDL}} \langle \pi \rangle \rho \wedge \langle \pi' \rangle \rho) \end{cases}$$

Therefore, by Lemma 1, $\langle \pi \cap \pi' \rangle \rho \leftrightarrow \langle \pi \rangle \rho \wedge \langle \pi' \rangle \rho$ is valid. (2.6):

$$\begin{split} w &\models_{\mathcal{C}\mathcal{GDL}} [\pi + \pi']\rho \\ = & \left\{ \text{ definition of } \models_{\mathcal{C}\mathcal{GDL}} \right\} \\ & \prod_{\substack{\phi \mid (w,\phi) \in [\pi + \pi'] \\ \phi \mid (w,\phi) \in [\pi + \pi'] \\ \forall \phi \mid (w,\phi) \in [\pi n] \\ \forall (w,\phi) \in [\pi n] \\ \forall (w,\phi) \in [\pi n] \\ & \prod_{\substack{\phi \mid (w,\phi) \in [\pi n] \\ \phi \mid (w,\phi) \in [\pi n] \\ (w,\phi) \in [\pi n]$$

(2.7):

$$\begin{split} &[\pi](\rho \wedge \rho') \\ &= \{ \text{ definition of } \models_{\mathcal{C}\mathcal{GDL}} \} \\ &\prod_{\phi \mid (w,\phi) \in [\![\pi]\!]} \left(\prod_{w' \in W} \left(\phi(w') \rightarrow w' \models_{\mathcal{C}\mathcal{GDL}} \rho \wedge \rho' \right) \right) \\ &= \{ \text{ definition of } \models_{\mathcal{C}\mathcal{GDL}} \} \\ &\prod_{\phi \mid (w,\phi) \in [\![\pi]\!]} \left(\prod_{w' \in W} \left(\phi(w') \rightarrow (w' \models_{\mathcal{C}\mathcal{GDL}} \rho \cdot w \models_{\mathcal{C}\mathcal{GDL}} \rho') \right) \right) \\ &\leq \{ \text{ definition of } \models_{\mathcal{C}\mathcal{GDL}} \} \\ &\prod_{\phi \mid (w,\phi) \in [\![\pi]\!]} \left(\prod_{w' \in W} \left((\phi(w') \rightarrow w' \models_{\mathcal{C}\mathcal{GDL}} \rho) \cdot (\phi(w') \rightarrow w' \models_{\mathcal{C}\mathcal{GDL}} \rho') \right) \right) \\ &= \{ (1) \} \\ &\prod_{\phi \mid (w,\phi) \in [\![\pi]\!]} \left(\prod_{w' \in W} \left(\phi(w') \rightarrow w' \models_{\mathcal{C}\mathcal{GDL}} \rho \right) \right) \\ &\cdot \prod_{\phi \mid (w,\phi) \in [\![\pi]\!]} \left(\prod_{w' \in W} \left(\phi(w') \rightarrow w' \models_{\mathcal{C}\mathcal{GDL}} \rho \right) \right) \\ &= \{ \text{ definition of } \models_{\mathcal{C}\mathcal{GDL}} \} \\ &(w \models_{\mathcal{C}\mathcal{GDL}} [\pi] \rho) \cdot (w \models_{\mathcal{C}\mathcal{GDL}} [\pi] \rho') \\ &= \{ \text{ definition of } \models_{\mathcal{C}\mathcal{GDL}} \} \\ &(w \models_{\mathcal{C}\mathcal{GDL}} [\pi] \rho \wedge [\pi] \rho') \end{split}$$