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Introduction to reversal fuzzy switch graph

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ABSTRACT

Fuzzy Switch Graphs (FSG) generalize the notion of Fuzzy Graphs by adding high-order arrows and aggregation functions which update the fuzzy values of arrows whenever a zero-order arrow is crossed. In this paper, we propose a more general structure called Reversal Fuzzy Switch Graph ($RFSG$), which promotes other actions in addition to updating the fuzzy values of the arrows, like activation and deactivation of the arrows. $RFSG$s are able to model dynamical aspects of some systems which generally appear in engineering, computer science and some other fields. The paper also provides the relationship between $RFSG$s and fuzzy graphs, a logic to verify properties of the modeled system and closes with an application.

1. Introduction

Reactive graphs are structures whose the relations change when we move along the graph. This concept has been introduced by Dov Gabbay in 2004 (see [12], [14]) and generalizes the static notion of a graph by incorporating high-order edges (called high-order arrows or switches). Graphs with these characteristics are called *Switch Graphs*.

In [22], Santiago et al. introduce the notion of *Fuzzy Switchs Graphs* (FSGs). These graphs are able to model reactive systems endowed with fuzziness and extend the notion of fuzzy graphs, in the sense that crossing an edge (zero-order arrow) induces an update of the system using high-order arrows and aggregation functions. For systems which require different aggregations for updating different arrows, Santiago et al. [22] introduced the *Fuzzy Reactive Graphs* (FRGs).

FSGs and FRGs, however, are not sufficient to model systems in which other edges of the system are activated or deactivated when one edge is crossing. To incorporate this, in [7] Campos, et al. propose the notion of *Reversal Fuzzy Switch Graphs* ($RFSG$s). Also in [7], the Cartesian product of $RFSG$s, a logic to verify properties of such structures and an application were presented. This paper complements reference [7] expanding its main contributions, presents an important relation between the $RFSG$s and fuzzy graphs and incorporates the logic notion of simulation and bisimulation.

The paper is organized as follows: Section 2 presents some basic concepts. Section 3 presents the notion of $RFSG$s, how they can be used to model the reactivity of some fuzzy systems and presents some algebraic operations. Section 4 provides a connection between the $RFSG$s and fuzzy graphs. Section 5 presents a logic and introduce the simulation and

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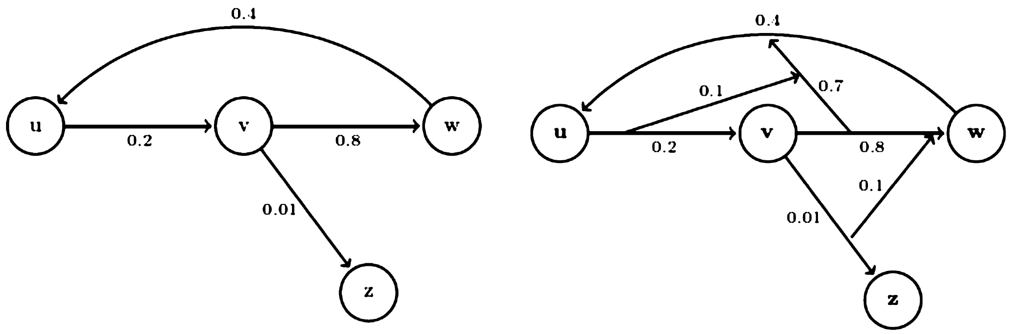


(a) Fuzzy graph   (b) Fuzzy switch graph

Fig. 1. Fuzzy graphs.

bisimulation for $RFSG$s. Section 6 shows how $RFSG$s can be used to model a dynamic control system. Finally, section 7 provides some final remarks.

2. Preliminaries

In this section we recall some concepts and results found in the literature in order to make this paper self-contained. We assume that the reader has some basic knowledge in fuzzy set theory. In order to make it easier to read, we will identify the membership function with the fuzzy set.

Definition 2.1. A **fuzzy set** A, defined on a non-empty set X, is characterized by a **membership function** $\varphi_A : X \to [0,1]$. The value $\varphi_A(x) \in [0,1]$ measures the degree of membership of x in the set A [16] [19].

Definition 2.2 *(Fuzzy Graphs [16] [20]).* A **fuzzy graph** is a structure $\langle V, R\rangle$, such that V is a non-empty set called **set of vertices** and R is a fuzzy set $R : V \times V \to [0,1]$.

For simplicity, we assume the set of vertices is a crisp set, in contrast to what is defined as a fuzzy graph in [16]. Fig. 1(a) shows a fuzzy graph.

Dov Gabbay [4] provided graphs with high-order arrows in order to model reactive behaviors. This kind of graphs is defined as follows.

Definition 2.3 *(Switch Graphs [4] [13]).* A **switch graph** is a pair $\langle W, R\rangle$ s.t. W is a non-empty set (set of **worlds**) and $R \subseteq A(W)$ is a set of arrows, called **switches** or **high-order arrows**, where $A(W) = \bigcup_{i\in\mathbb{N}} A_i(W)$ with

$$\begin{cases} A_0(W) = W \times W \\ A_{i+1}(W) = A_0(W) \times A_i(W) \end{cases} \tag{1}$$

Fuzzy Switch Graphs were introduced by Santiago et al. in [22].

Definition 2.4 *(Fuzzy Switch Graphs [22]).* Let W be a non-empty finite set (set of **states** or **worlds**) and the family of sets $S = \bigcup_{n\in\mathbb{N}} S^n$ where $S^0 \neq \emptyset$ and

$$\begin{cases} S^0 \subseteq W \times W \\ S^{n+1} \subseteq S^0 \times S^n \end{cases} \tag{2}$$

A **fuzzy switch graph (FSG)** is a pair $\mathcal{M} = \langle W, \varphi : S \to [0,1]\rangle$, where φ is a fuzzy set on S. The elements $a_i^0 \in S^0$ $(i \in \mathbb{N})$ are called **zero-order arrows**. The elements of S^{n+1} are called **high-order arrows**.

Example 2.1. Fig. 1 shows a fuzzy graph and a fuzzy switch graph.

Fuzzy Logic provides many proposals for logical connectives. In what follows we review the notions of t-norms, t-conorms, fuzzy implications and fuzzy negations. The first two cases are generalizations of the classic notion of disjunctions and conjunctions, respectively [15].

Definition 2.5 *(t-norms and t-conorms).* A **uninorm** is a bivariate function $U : [0,1] \times [0,1] \to [0,1]$, that is isotonic, commutative, associative with a neutral element $e \in [0,1]$. If $e = 1$, then U is called **t-norm** and if $e = 0$, then U is called **t-conorm**.

Example 2.2. The functions $T_G(x, y) = min(x, y)$ and $T_L(x, y) = max(x + y - 1, 0)$ (*Łukasiewicz*) are t-norms. The functions $S_G(x, y) = max(x, y)$ and $S_L(x, y) = min(x + y, 1)$ (*Łukasiewicz*) are t-conorms.

Notation 1*:* Let T be a t-norm, $f : [0,1] \to [0,1]$ and J_n a finite subset of $[0,1]$ with n elements ($J_0 = \emptyset$). We define $\underset{a\in J_n}{T}$,

$$\underset{a\in J_n}{T} f(a) = \begin{cases} 1, \text{ case } n = 0; \\ f(a), \text{ case } n = 1; \\ T\big(f(x), \underset{a\in J_m}{T} f(a)\big), \text{ case } n > 1, x \in J_n \text{ and } J_m = J_n \backslash \{x\}. \end{cases} \tag{3}$$

Similarly, for S t-conorm, we define $\underset{a\in J_n}{S}$ s.t.

$$\underset{a\in J_n}{S} f(a) = \begin{cases} 0, \text{ case } n = 0; \\ f(a), \text{ case } n = 1; \\ S\big(f(x), \underset{a\in J_m}{S} f(a)\big), \text{ case } n > 1, x \in J_n \text{ and } J_m = J_n \backslash \{x\}. \end{cases} \tag{4}$$

Note that, since T and S are commutative and associative, $\underset{a\in J_m}{T}$ and $\underset{a\in J_m}{S}$ are well defined. That is, it does not depend on the way we choose $x \in J_n$ to make $J_n = \{x\} \cup J_m$.

Example 2.3. Given the t-norm $T(x, y) = min(x, y)$, the identity function $Id : [0,1] \to [0,1]$ and the set $J_3 = \{x_1, x_2, x_3\} \subset [0,1]$, we have:

$$\begin{aligned} \underset{a\in J_3}{T} Id(a) &= min\left(x_1, \underset{a\in J_2}{T} Id(a)\right) \\ &= min\left(x_1, min\big(x_2, \underset{a\in J_1}{T} Id(a)\big)\right) \\ &= min\left(x_1, min\big(x_2, Id(x_3)\big)\right) \\ &= min\left(x_1, min\big(x_2, x_3\big)\right). \end{aligned}$$

Definition 2.6 *(Negations [3], [21]).* A unary operation $N : [0,1] \to [0,1]$ is a **fuzzy negation** if $N(0) = 1, N(1) = 0$ and N is decreasing.

Example 2.4. *Gödel Negation*: $N_G : [0,1] \to [0,1]$ s.t. $N_G(0) = 1$ and $N_G(x) = 0$, whenever $x > 0$.

Definition 2.7 *(Implications [3]).* A bivariate function $I : [0,1]^2 \to [0,1]$ is a **fuzzy implication** if it is decreasing with respect to the first variable, increasing with respect to the second variable, $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$ (boundary conditions).

Example 2.5. *Gödel Implication*: $I_G : [0,1]^2 \to [0,1]$ s.t. $I_G(x, y) = 1$, whenever $x \le y$, and $I_G(x, y) = y$ otherwise.

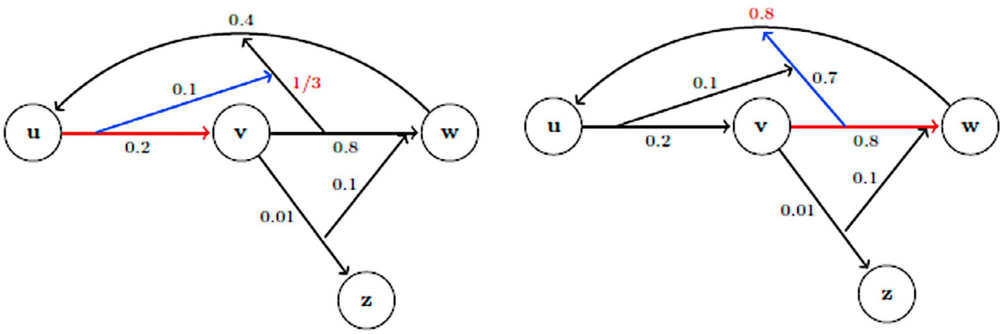
Definition 2.8 *(Bi-implications [6]).* A bivariate function $B : [0,1]^2 \to [0,1]$ is a **fuzzy bi-implication** if it is commutative, $B(x, x) = 1$, $B(0,1) = 0$ and $B(w, z) \le B(x, y)$, whenever $w \le x \le y \le z$.

Example 2.6. *Gödel Bi-implication*: $B_G(x, y) = T_G(I_G(x, y), I_G(y, x))$.

Definition 2.9 *(Fuzzy Semantics [9]).* A structure $\mathcal{F} = \{[0,1], T, S, N, I, B, 0, 1\}$, s.t. T is a t-norm, S is a t-conorm, N is a fuzzy negation, I is a fuzzy implication and B is a fuzzy bi-implication, is called a **fuzzy semantics**.

Example 2.7. *Gödel Semantic*: $\mathcal{G} = \{[0,1], T_M, S_M, N_G, I_G, B_G, 0, 1\}$.

Aggregation functions [18], [8], [1], [2], [10] are functions with special properties which generalize the means, like *arithmetic mean*, *weighted mean* and *geometric mean*.



(a) $\mathcal{M}_R$ after crossing $a_1^0 = (u, v)$ (b) $\mathcal{M}_R$ after crossing $a_3^0 = (v, w)$

Fig. 2. Reconfigurations of $\mathcal{M}_R$.

Definition 2.10 *(Aggregation Function [5]).* An **aggregation function** is a n-ary function $A : [0,1]^n \to [0,1]$, with $A(0,0,...,0) = 0$, $A(1,1,...,1) = 1$ and, for all $\bar{x}, \bar{y} \in [0,1]^n$, $\bar{x} \leq \bar{y}$ implies $A(\bar{x}) \leq A(\bar{y})$.

Example 2.8. $A_n(\bar{x}) = \dfrac{1}{n}(x_1 + ... + x_n)$ (*Arithmetic mean*), $A_n(\bar{x}) = \sqrt[n]{x_1 \cdot ... \cdot x_n}$ (*Geometric mean*), t-norms, t-conorms and projection functions, $\Pi_j : A_1 \times ... \times A_j \times ... \times A_n \longrightarrow A_j$, s.t. $\Pi_j(x_1, ..., x_j, ..., x_n) = x_j$, are aggregation functions.

Definition 2.11 *([5]).* For every $\bar{x} \in [0,1]^n$, an aggregation function A is, **average** if $min(\bar{x}) \leq A(\bar{x}) \leq max(\bar{x})$, **conjunctive** if $A(\bar{x}) \leq min(\bar{x})$ and **disjunctive** if $A(\bar{x}) \geq max(\bar{x})$.

Example 2.9. t-norms are conjunctive aggregations, t-conorms are disjunctive and means (*arithmetic, geometric, weighted*) are average aggregations. For example, given $x, y \in [0,1]$ we have:

$$xy \leq min\{x, y\} \leq \frac{x+y}{2} \leq max\{x, y\}.$$

Definition 2.12. An aggregation $A : [0,1]^n \to [0,1]$ is **shift-invariant** if, for all $\lambda \in [-1,1]$ and for all $(x_1, ..., x_n) \in [0,1]^n$,

$$A(x_1 + \lambda, ..., x_n + \lambda) = A(x_1, ..., x_n) + \lambda$$

whenever $(x_1 + \lambda, ..., x_n + \lambda) \in [0,1]^n$ and $A(x_1, ..., x_n) + \lambda \in [0,1]$.

In [22], Santiago et al. extend the notion of FSGs for *Fuzzy Reactive Graphs*. In what follows, given a FSG $\mathcal{M} = \langle W, \varphi : S \to [0,1] \rangle$, we define the set $S_{\to} = \{a_i^0 \in S^0; [\![a_i^0, a]\!] \in S,\ \text{with } a \in S\}$.

Definition 2.13 *(Fuzzy Reactive Graphs).* Let $\mathcal{M} = \langle W, \varphi : S \to [0,1] \rangle$ be a FSG, $A_{\mathcal{M}} = \{A_1, ..., A_k : [0,1]^3 \to [0,1]\}$ a set of aggregation functions and a function $Ag_{\mathcal{M}} : S_{\to} \to A_{\mathcal{M}}$. The pair $\mathcal{M}_R = \langle \mathcal{M}, Ag_{\mathcal{M}} \rangle$ is called a **fuzzy reactive graph (FRG)**.

Notation 2*:* In order to make the presentation of the graphs and the movements on the graph more intuitive, we will establish: Arrows that are crossed over the graph will be drawn in red. High-order arrows that act on the graph configuration, after crossing the zero-order arrow, will be drawn in blue. The first arrow crossed will have a single point, the second arrow crossed will have a double point, the third arrow to be crossed will have a triple point and so on. If multiple movements are made repeatedly on the same arrow, the arrow pointer will show the order of the last movement. For example, if the movement is made three times on the same arrow, graphically, we will see only a red triple-headed arrow in the graph.

Example 2.10. Let $\mathcal{M}$ be the FSG in Fig. 1(b). Consider $S^0 = \{a_1^0 = (u, v), a_2^0 = (v, v), a_3^0 = (v, w), a_4^0 = (v, z), a_5^0 = (w, u)\}$ and $A_{\mathcal{M}} = \{arith,\ max\}$. Defining the application $Ag_{\mathcal{M}} : S_{\to} \to A_{\mathcal{M}}$ s.t. $Ag(a_1^0) = Ag(a_2^0) = arith$ and $Ag(a_3^0) = Ag(a_4^0) = Ag(a_5^0) = max$, we have the FRG $\mathcal{M}_R = \langle \mathcal{M}, Ag \rangle$. Fig. 2(a) contains the reconfiguration of $\mathcal{M}_R$ after crossing $a_1^0 = (u, v)$ and having applied $Ag(a_1^0) = arith$ to the fuzzy values: 0.2, 0.1, 0.7. We calculate $arith(0.2, 0.1, 0.7) = (0.2 + 0.1 + 0.7)/3 = 1/3$ and the arrow $[\![[vw], [wu]]\!]$ gets the new fuzzy value 1/3. Fig. 2(b) contains the reconfiguration of $\mathcal{M}_R$ after crossing $a_3^0 = (v, w)$ and having applied $Ag(a_3^0) = max$ to the fuzzy values: 0.8, 0.7, 0.4. We calculate $max(0.8, 0.7, 0.4) = 0.8$ and the arrow $[wu]$ gets the new fuzzy value 0.8.

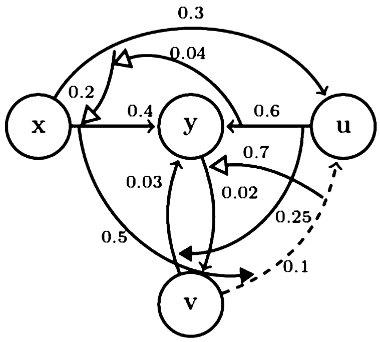


Fig. 3. Reversal fuzzy switch graph ($RFSG$).

3. Reversal fuzzy switch graph

In this section we introduce the notion of Reversal Fuzzy Switch Graph, a structure which generalizes the notion of Fuzzy Switch Graph introduced by Santiago et al. [22]. This new kind of graph has in its structure two new types of high order-arrows, called connecting arrows and disconnecting arrows. These arrows allow to model reactive systems in which the accessibility to the worlds may be activated or deactivated by the transitions.

In what follows W and V are non-empty finite sets.

Definition 3.1 *(Reversal Fuzzy Switch Graphs [7]).* Let W be a set whose elements are called **states** or **worlds**. Consider the following family of sets defined recursively,

$$\begin{cases} S^0 \subseteq W \times W \\ S^{n+1} \subseteq S^0 \times S^n \times \{\bullet, \circ\} \end{cases} \tag{5}$$

s.t. $S^0 \neq \emptyset$ and for any $n \geq 1$, $(a_i^0, a, \circ) \notin S^n$ or $(a_i^0, a, \bullet) \notin S^n$. Consider $S = \bigcup_{n \in \mathbb{N}} S^n$, a **reversal fuzzy switch graph (RFSG) is a pair** $M = \langle W, \mu : S \to [0,1] \times \{\textsc{on}, \textsc{off}\}\rangle$.[1] Arrows with $\bullet$ in their third component are called **connecting arrows** and arrows with $\circ$ in their third component are called **disconnecting arrows**. When the context is clear we denote a $RFSG$ simply by $\langle W, \mu \rangle$.

Active arrows are drawn with a normal line whereas inactive arrows are drawn with a dashed line. Moreover, connecting (disconnecting) arrows change the targeted arrow state for active (inactive) and are drawn with a black (white) arrowhead.

For readability, we introduce some notation and nomenclatures:

- Arrows in S^n will be denoted by a_i^n, for $n \geq 0$ and $i \in \mathbb{N}$.
- In the following, we make an abuse of notation. When necessary and if the context is clear, we will denote in more detail the arrows in S^n in a more expanded way. For example, a_i^0 from x to y will be denoted by $[xy]$, the disconnecting and connecting first-order arrows, from $[xy]$ to $[uv]$ will be denoted by $[\![[xy],[uv],\circ]\!]$ and $[\![[xy],[uv],\bullet]\!]$, respectively. When referring to any high-order arrow, we write $\sigma \in \{\circ, \bullet\}$ instead of $\circ$ or $\bullet$. For example, any first-order arrow from $[uv]$ to $[xy]$ will be written $[\![[uv],[xy],\sigma]\!]$. Any second-order arrows from $[zw]$ to $[\![[xy],[uv],\sigma]\!]$ will be denoted by $[\![[zw],[\![[xy],[uv],\sigma]\!],\sigma']\!]$.
- When there is no need to specify the order of the arrow belonging to set S, we will denote $a \in S$.
- Given the projection functions $\Pi_1 : [0,1] \times \{\textsc{on}, \textsc{off}\} \to [0,1]$ and $\Pi_2 : [0,1] \times \{\textsc{on}, \textsc{off}\} \to \{\textsc{on}, \textsc{off}\}$, if $a \in S$ we write $\mu_1(a) = \Pi_1(\mu(a))$ and $\mu_2(a) = \Pi_2(\mu(a))$ to indicate the first and second components of $\mu(a)$.
- Let $R \subseteq S$, the set of active arrows in R is denoted by

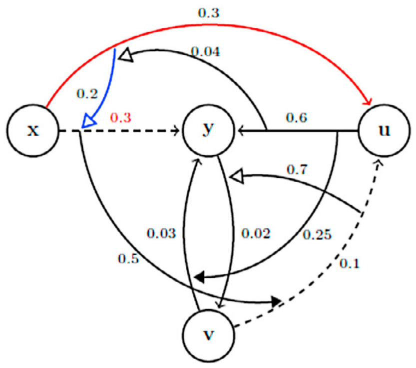
  $$R_\mu^* := \{a \in R;\ \mu_2(a) = \textsc{on}\}$$

  and the set of arrows in R which is the origin of a high-order arrow in S is denoted by

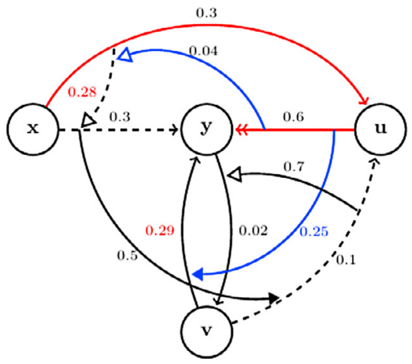
  $$R_\to = \left\{a_i^0 \in R;\ [\![a_i^0, b, \sigma]\!] \in S \text{ with } b \in S \text{ and } \sigma \in \{\circ, \bullet\}\right\}.$$

In the following, we will consider the $RFSG$s $M = \langle W, \mu \rangle$ and $M' = \langle W, \mu' \rangle$.

[1] In this paper we assume that the membership function is valued in the complete lattice $[0.1] \times \{\textsc{on}, \textsc{off}\}$ using the product order where $\textsc{off} \leq \textsc{on}$.



(a) $M^{arith}_{[xu]}$



(b) $M^{arith}_{[xu][uy]}$

Fig. 4. Reactivity of $RFSG$ after crossing zero-order arrows $[xu]$ and $[xu][uy]$.

Definition 3.2. M is a **subgraph** of M' if $\mu_1(a) \le \mu'_1(a)$ and $\mu_2(a) = \mu'_2(a)$, for all $a \in S$. M is a **supergraph** of M' if $\mu_1(a) \ge \mu'_1(a)$ and $\mu_2(a) = \mu'_2(a)$, for all $a \in S$.

Definition 3.3. M' is a **translation** of M by $\lambda \in [-1, 1]$ if, for all $a \in S$ s.t. $\mu_1(a) > 0$, $\mu'_1(a) = \mu_1(a) + \lambda \in [0, 1]$ and $\mu'_2(a) = \mu_2(a)$.

3.1. Reactivity of $RFSGs$

Intuitively, a reactive graph is a graph that may change its configuration when a zero-order arrow is crossed. In order to model this global dependence in a $RFSG$, we consider the reactivity idea presented in [22] with the necessary adaptations: *Whenever a zero-order arrow is crossed, the fuzzy value and the arrow state (active or inactive) of its target arrows are updated.*

Definition 3.4. Given a $RFSG$ $M = \langle W, \mu\rangle$ with aggregation function $A : [0, 1]^3 \to [0, 1]$, a $RFSG$ **based on** A **after crossing an active zero-order arrow** a^0_i, is the $RFSG$ $M^A_{a^0_i} = \langle W, \mu^A_{a^0_i} : S \to [0, 1] \times \{\text{ON}, \text{OFF}\}\rangle$ s.t.

$$\mu^A_{a^0_i}(a) = \begin{cases} \Big(A\Big(\mu_1(a^0_i), \mu_1([\![a^0_i, a, \bullet]\!]), \mu_1(a)\Big), \text{ON}\Big), \; if \; [\![a^0_i, a, \bullet]\!] \in S^*_\mu; \\ \Big(A\Big(\mu_1(a^0_i), \mu_1([\![a^0_i, a, \circ]\!]), \mu_1(a)\Big), \text{OFF}\Big), \; if \; [\![a^0_i, a, \circ]\!] \in S^*_\mu; \\ \mu(a), \; otherwise. \end{cases} \tag{6}$$

The $RFSG$ $M^A_{a^0_i}$ is called **reconfiguration of** M**, based on** A**, after crossing** a^0_i.

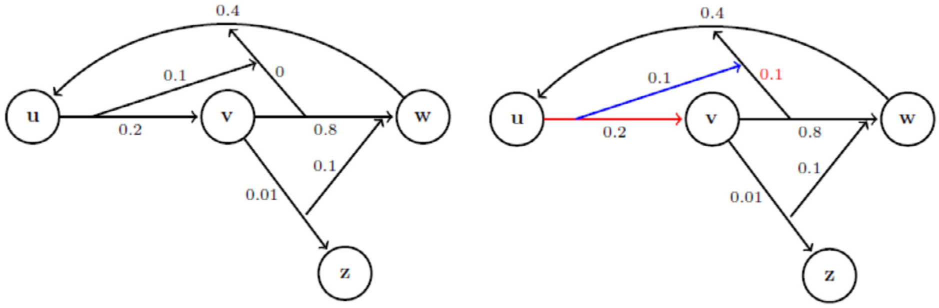
Let us see how this definition works in Fig. 4 using the *arithmetic mean* as aggregation function after crossing a sequence of zero-order arrows in Fig. 3. After the arrow $a^0_1 = [xu]$ has been crossed, Fig. 4(a), the arrow $a^0_2 = [xy]$ is *updated* due to $a^1_1 = [\![[xu], [xy], \circ]\!]$ by the *arithmetic mean* between the fuzzy values $\mu_1(a^0_1)$, $\mu_1(a^0_2)$ and $\mu_1(a^1_1)$, and by replacing the marker ON to OFF (the arrow a^0_2 becomes inactive). In a second step and in the same manner, after the arrow $a^0_3 = [uy]$ has been crossed, the arrow $a^1_1 = [\![[xu], [xy], \circ]\!]$ has its fuzzy value updated and becomes inactive, however, the arrow $a^0_5 = [vy]$ has only its fuzzy value updated since it is an active arrow targeted by a connecting arrow (Fig. 4(b)).

Remark 3.1. The edges contained in S can receive a null fuzzy value. However, graphically, these arrows will be displayed only if there is the possibility of modifying this value by some high-order arrow (Fig. 5).

From the action of an aggregation, after a reconfiguration, the value of an arrow with a non-null fuzzy value can be modified until this value is zero.

Proposition 3.1. *If A is a conjunctive (disjunctive) aggregation and $(\mu^A_{a^0_i})_2(b) = \mu_2(b)$ for all $b \in S$, then $M^A_{a^0_i}$ is a subgraph (supergraph) of M.*

Proof. Given $b \in S$ and denoting $\Big(\mu_1(a^0_i), \mu_1([\![a^0_i, b, \sigma]\!]), \mu_1(b)\Big) = \overline{[\![a^0_i, b, \sigma]\!]}$:



(a) M (b) $M^{arith}_{[uv]}$

Fig. 5. $RFSG$ with a null fuzzy value.

- Case $[\![a_i^0, b, \sigma]\!] \in S_\mu^*$:

$$\left(\mu^A_{a_i^0}\right)_1(b) = A\left(\overline{[\![a_i^0, b, \sigma]\!]}\right) \leq \min\left(\overline{[\![a_i^0, b, \sigma]\!]}\right) \leq \mu_1(b).$$

- Case $[\![a_i^0, b, \sigma]\!] \notin S_\mu^*$:

$$\left(\mu^A_{a_i^0}\right)_1(b) = \mu_1(b).$$

Then $M^A_{a_i^0}$ is subgraph of M. The dual statement follows straightforwardly.

Proposition 3.2. *Let M' be a translation of M by $\lambda \in [-1, 1]$. If A is shift-invariant, then $M'^A_{a_i^0}$ is a translation of $M^A_{a_i^0}$ by λ.*

Proof. Let $b \in S$. Denoting $\left(\mu'_1(a_i^0), \mu'_1([\![a_i^0, b, \sigma]\!]), \mu'_1(b)\right) = \overline{[\![a_i^0, b, \sigma]\!]}$ and supposing that $A\Big(\mu_1(a_i^0), \mu_1([\![a_i^0, b, \sigma]\!]), \mu_1(b)\Big) + \lambda \in [0, 1]$:

- Case $[\![a_i^0, b, \sigma]\!] \in S_\mu^*$:

$$\begin{aligned}\left(\mu'^A_{a_i^0}\right)_1(b) &= A\left(\overline{[\![a_i^0, b, \sigma]\!]}\right) \\ &= A\left(\mu_1(a_i^0) + \lambda, \mu_1([\![a_i^0, b, \sigma]\!]) + \lambda, \mu_1(b) + \lambda\right) \\ &= A\left(\mu_1(a_i^0), \mu_1([\![a_i^0, b, \sigma]\!]), \mu_1(b)\right) + \lambda \\ &= \left(\mu^A_{a_i^0}\right)_1(b) + \lambda\end{aligned}$$

- Case $[\![a_i^0, b, \sigma]\!] \notin S_\mu^*$:

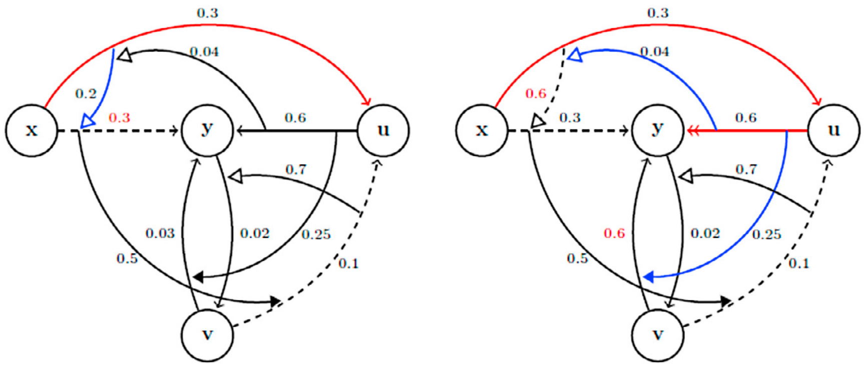
$$\left(\mu'^A_{a_i^0}\right)_1(b) = \mu'_1(b) = \mu_1(b) + \lambda = \left(\mu^A_{a_i^0}\right)_1(b) + \lambda.$$

By hypotheses, $\mu_2(b) = \mu'_2(b)$, then $\left(\mu'^A_{a_i^0}\right)_2(b) = \left(\mu^A_{a_i^0}\right)_2(b)$.

Next, we will provide an extension for the notion of reactivity presented in [22]. Just as it is done for the case of FRGs, each active zero-order arrow triggers an aggregation function.

Definition 3.5 *(Reversal Fuzzy Reactive Graphs [7]).* Consider M a $RFSG$, $A = \{A_1, ..., A_k : [0, 1]^3 \to [0, 1]\}$ a set of aggregation functions and a function $A_g : S_\to \to A$. The pair $M_R = \langle M, Ag\rangle$ is called **reversal fuzzy reactive graph (RFRG)**.

If $a_i^0 \in S_\mu{}^{0*}$, the **reconfiguration of M_R after crossing a_i^0** is the $RFRG$ $M_R^{a_i^0} = \langle M^{a_i^0}, A_g\rangle$, where $M^{a_i^0} = \langle W, \mu^{A_g}_{a_i^0}\rangle$ is a $RFSG$ s.t.



(a) $M_R^{[xu]}$ (b) $M_R^{[xu][uy]}$

Fig. 6. Reconfigurations of M_R.

Table 1
Differences between fuzzy graphs.

	zero-order arrows	high-order arrows	connection or disconnection high-order arrows	one aggregation associated	more than one aggregation associated
FG	✓				
FSG	✓	✓		✓	
FRG	✓	✓			✓
$RFSG$	✓		✓	✓	
$RFRG$	✓		✓		✓

$$\mu_{a_i^0}^{A_g}(b) = \begin{cases} \Big(A_g(a_i^0)\Big(\mu_1(a_i^0), \mu_1([\![a_i^0, b, \bullet]\!]), \mu_1(b)\Big), \textsc{on}\Big), & if\ [\![a_i^0, b, \bullet]\!] \in S_\mu^*; \\ \Big(A_g(a_i^0)\Big(\mu_1(a_i^0), \mu_1([\![a_i^0, b, \circ]\!]), \mu_1(b)\Big), \textsc{off}\Big), & if\ [\![a_i^0, b, \circ]\!] \in S_\mu^*; \\ \mu(b), \ otherwise. \end{cases} \tag{7}$$

Example 3.1. Let M be the $RFSG$ at Fig. 3, $S^0 = \{[xy], [xu], [uy], [vy], [yv], [vu]\}$, $A = \{arith, max\}$, $A_g([xy]) = A_g([xu]) = A_g([yv]) = arith$ and $A_g([vy]) = A_g([uy]) = A_g([vu]) = max$. Fig. 6 contains $M_R^{[xu]}$ and $M_R^{[xu][uy]}$, respectively.

At this point, in order to clearly expose the differences between the different fuzzy graphs presented here, we present the Table 1.

3.2. Product of $RFSG$s

In the following, we will consider the $RFSG$s $M = \langle W, \mu : S \to [0,1] \times \{\textsc{on}, \textsc{off}\}\rangle$ and $N = \langle V, \delta : T \to [0,1] \times \{\textsc{on}, \textsc{off}\}\rangle$ with W and V disjoint set; and the set $W \star V = \bigcup_{n \in \mathbb{N}} (W \star V)^n$ s.t.,

$$\begin{cases} (W \star V)^0 \subseteq (W \times V) \times (W \times V) \\ (W \star V)^{n+1} \subseteq (W \star V)^0 \times (W \star V)^n \times \{\bullet, \circ\}. \end{cases}$$

Given $a_i^0 \in (W \star V)^0$, $a \in (W \star V)^n$ and $\sigma \in \{\circ, \bullet\}$, we will consider the subsets

- $(W \star V)_S^0 = \big\{[(w_i, v)(w_j, v)] \in (W \star V)^0 \ ; \ v \in V \text{ and } [w_i w_j] \in S^0\big\}$,
- $(W \star V)_T^0 = \big\{[(w, v_i)(w, v_j)] \in (W \star V)^0 \ ; \ w \in W \text{ and } [v_i v_j] \in T^0\big\}$,
- $(W \star V)_S^{n+1} = \big\{[\![a_i^0, a, \sigma]\!] \in (W \star V)^{n+1} \ ; \ a_i^0 \in (W \star V)_S^0 \text{ and } a \in (W \star V)_S^n\big\}$,
- $(W \star V)_T^{n+1} = \big\{[\![a_i^0, a, \sigma]\!] \in (W \star V)^{n+1} \ ; \ a_i^0 \in (W \star V)_T^0 \text{ and } a \in (W \star V)_T^n\big\}$,

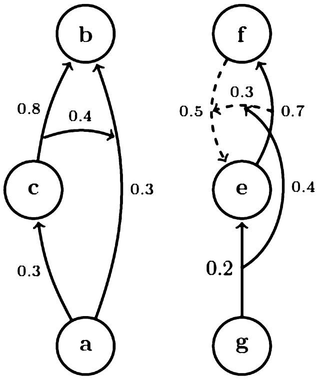


Fig. 7. $RFSG$s M and N.

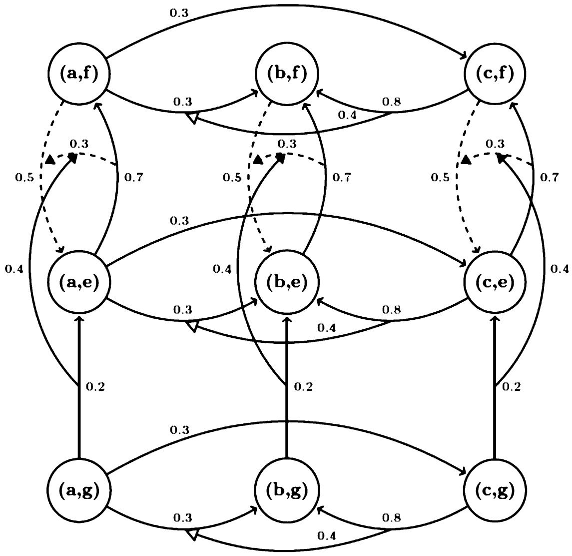


Fig. 8. Cartesian product $M \times N$.

and the application $\xi : (W \star V)_{S \cup T} = \bigcup_{n \in \mathbb{N}} \left[(W \star V)^n_S \cup (W \star V)^n_T\right] \to S \cup T$ s.t.

$$\xi(b) = \begin{cases} [w_i w_j] \in S^0, \text{ if } b = [(w_i, v)(w_j, v)] \in (W \star V)^0_S; \\ [v_i v_j] \in T^0, \text{ if } b = [(w, v_i)(w, v_j)] \in (W \star V)^0_T; \\ [\![\xi(a^0_i), \xi(a), \sigma]\!] \in S^{n+1}, \text{ if } b = [\![a^0_i, a, \sigma]\!] \in (W \star V)^{n+1}_S; \\ [\![\xi(a^0_i), \xi(a), \sigma]\!] \in T^{n+1}, \text{ if } b = [\![a^0_i, a, \sigma]\!] \in (W \star V)^{n+1}_T. \end{cases}$$

Definition 3.6 *(Product of $RFSG$s).* The **Cartesian Product** of the $RFSG$s M and N is the $RFSG$: $M \times N = \langle W \times V\ ,\ \psi : (W \star V)_{S \cup T} \to [0,1] \times \{\text{ON}, \text{OFF}\}\rangle$ s.t.

$$\psi(b) = \begin{cases} \mu\big(\xi(b)\big), \text{ if } b \in (W \star V)_S \\ \delta\big(\xi(b)\big), \text{ if } b \in (W \star V)_T \end{cases} \tag{8}$$

Example 3.2. Consider M and N shown in Fig. 7. The product $M \times N$ can be observed in Fig. 8.

In order to define the product of $RFRG$s, we consider:

- The $RFRG$s $M_R = \langle M, Ag_M\rangle$ and $N_R = \langle N, Ag_N\rangle$;
- The functions $Ag_M : S_\to \to A_M$ and $Ag_N : T_\to \to A_N$;
- The sets of aggregations A_M and A_N;

The aggregations $a_m \in A_M$ and $a_n \in A_N$ will be denoted by $(M, a_m) : [0,1]^3 \to [0,1]$ and $(N, a_n) : [0,1]^3 \to [0,1]$.

Definition 3.7 *(Product of RFRGs).* Consider the $RFRG$s M_R and N_R, the set $A_M \oplus A_N = \{(M, a_m) : a_m \in A_M\} \cup \{(N, a_n) : a_n \in A_N\}$ and the function $Ag_{M\times N} : [(W \star V)^0_S \cup (W \star V)^0_T]_{\to} \to A_M \oplus A_N$ s.t.

$$Ag_{M\times N}(a_i^0) = \begin{cases} \Big(N, Ag_N\big(\xi(a_i^0)\big)\Big), & \text{if } a_i^0 \in (W \star V)^0_T \\ \Big(M, Ag_M\big(\xi(a_i^0)\big)\Big), & \text{if } a_i^0 \in (W \star V)^0_S \end{cases} \tag{9}$$

The structure $M_R \times N_R = \langle M \times N, Ag_{M\times N}\rangle$ is the **product of** $RFRG$**s** M_R **and** N_R.

The next proposition ensures that the updated product is obtained from the updated factors.

Proposition 3.3. *Consider the $RFRG$s M_R, N_R, the product $M_R \times N_R$, $a_i^0 \in (W \star V)^0_S \cup (W \star V)^0_T$ and $a \in (W \star V)_S \cup (W \star V)_T$ s.t. $\Big(\psi_1(a_i^0), \psi_1(\llbracket a_i^0, a, \circ\rrbracket), \psi_1(a)\Big) = \overline{\llbracket a_i^0, a, \circ\rrbracket}$ and $\Big(\psi_1(a_i^0), \psi_1(\llbracket a_i^0, a, \bullet\rrbracket), \psi_1(a)\Big) = \overline{\llbracket a_i^0, a, \bullet\rrbracket}$. Then,*

$$\psi^{Ag_{M\times N}}_{a_i^0}(a) = \begin{cases} \delta\big(\xi(a)\big), \ \text{if } C_1; \\ \mu\big(\xi(a)\big), \ \text{if } C_2; \\ \Big(Ag_N(\xi(a))\big(\overline{\llbracket a_i^0, a, \circ\rrbracket}\big), \text{OFF}\Big), \ \text{if } C_3; \\ \Big(Ag_N(\xi(a))\big(\overline{\llbracket a_i^0, a, \bullet\rrbracket}\big), \text{ON}\Big), \ \text{if } C_4; \\ \Big(Ag_M(\xi(a))\big(\overline{\llbracket a_i^0, a, \circ\rrbracket}\big), \text{OFF}\Big), \ \text{if } C_5; \\ \Big(Ag_M(\xi(a))\big(\overline{\llbracket a_i^0, a, \bullet\rrbracket}\big), \text{ON}\Big), \ \text{if } C_6; \end{cases} \tag{10}$$

For:

- $C_1 : a \in (W \star V)_T$ *and* $\llbracket a_i^0, a, \sigma\rrbracket \notin [(W \star V)_S \cup (W \star V)_T]^*_\psi$;
- $C_2 : a \in (W \star V)_S$ *and* $\llbracket a_i^0, a, \sigma\rrbracket \notin [(W \star V)_S \cup (W \star V)_T]^*_\psi$;
- $C_3 : a_i^0 \in (W \star V)^0_T$ *and* $\llbracket a_i^0, a, \circ\rrbracket \in [(W \star V)_S \cup (W \star V)_T]^*_\psi$;
- $C_4 : a_i^0 \in (W \star V)^0_T$ *and* $\llbracket a_i^0, a, \bullet\rrbracket \in [(W \star V)_S \cup (W \star V)_T]^*_\psi$;
- $C_5 : a_i^0 \in (W \star V)^0_S$ *and* $\llbracket a_i^0, a, \circ\rrbracket \in [(W \star V)_S \cup (W \star V)_T]^*_\psi$;
- $C_6 : a_i^0 \in (W \star V)^0_S$ *and* $\llbracket a_i^0, a, \bullet\rrbracket \in [(W \star V)_S \cup (W \star V)_T]^*_\psi$.

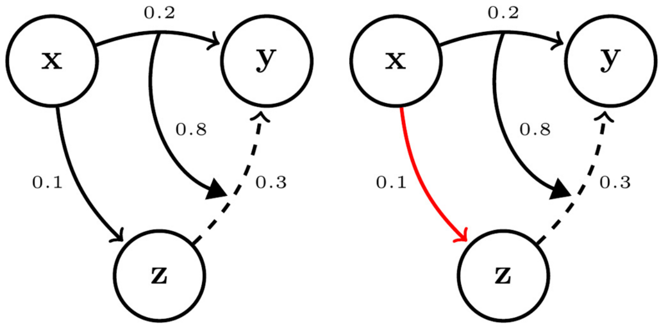
Proof. Indeed,

- Case $\llbracket a_i^0, a, \sigma\rrbracket \notin [(W \star V)_S \cup (W \star V)_T]^*_\psi$,
  Case $a \in (W \star V)_T$: $\psi^{Ag_{M\times N}}_{a_i^0}(a) \stackrel{\text{def}}{=} \psi(a) \stackrel{\text{def}}{=} \delta(\xi(a))$.
  Case $a \in (W \star V)_S$: $\psi^{Ag_{M\times N}}_{a_i^0}(a) \stackrel{\text{def}}{=} \psi(a) \stackrel{\text{def}}{=} \mu(\xi(a))$.
- Case $\llbracket a_i^0, a, \circ\rrbracket \in [(W \star V)_S \cup (W \star V)_T]^*_\psi$,
  Case $a \in (W \star V)_T$:

$$\psi^{Ag_{M\times N}}_{a_i^0}(a) \stackrel{\text{def}}{=} \Big(Ag_{M\times N}(a_i^0)\big(\overline{\llbracket a_i^0, a, \circ\rrbracket}\big), \text{OFF}\Big) \stackrel{\text{def}}{=} \Big((N, Ag_N)(\xi(a))\big(\overline{\llbracket a_i^0, a, \circ\rrbracket}\big), \text{OFF}\Big)$$
$$\stackrel{\text{def}}{=} \Big(Ag_N(\xi(a))\big(\overline{\llbracket a_i^0, a, \circ\rrbracket}\big), \text{OFF}\Big).$$

  Case $a \in (W \star V)_S$:

$$\psi^{Ag_{M\times N}}_{a_i^0}(a) \stackrel{\text{def}}{=} \Big(Ag_{M\times N}\big(a_i^0\big)\big(\overline{\llbracket a_i^0, a, \circ\rrbracket}\big), \text{OFF}\Big) \stackrel{\text{def}}{=} \Big((M, Ag_M)(\xi(a))\big(\overline{\llbracket a_i^0, a, \circ\rrbracket}\big), \text{OFF}\Big)$$
$$\stackrel{\text{def}}{=} \Big(Ag_M(\xi(a))\big(\overline{\llbracket a_i^0, a, \circ\rrbracket}\big), \text{OFF}\Big).$$



(a) μ (b) $\mu^A_{[xy]}$

Fig. 9. $RFSG$s M and $M^A_{[xz]}$.

- Case $[\![a_i^0, a, \bullet]\!] \in [(W \star V)_S \cup (W \star V)_T]^*_{\psi}$,
  Case $a \in (W \star V)_T$:

$$\psi_{a_i^0}^{Ag_{M\times N}}(a) \stackrel{\text{def}}{=} \left(Ag_{M\times N}(a_i^0)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right) \stackrel{\text{def}}{=} \left((N, Ag_N)(\xi(a))(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right)$$
$$\stackrel{\text{def}}{=} \left(Ag_N(\xi(a))(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right).$$

  Case $a \in (W \star V)_S$:

$$\psi_{a_i^0}^{Ag_{M\times N}}(a) \stackrel{\text{def}}{=} \left(Ag_{M\times N}(a_i^0)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right) \stackrel{\text{def}}{=} \left((M, Ag_M)(\xi(a))(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right)$$
$$\stackrel{\text{def}}{=} \left(Ag_M(\xi(a))(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right).$$

4. *RFSG*s and fuzzy graphs

In this section, given an $RFSG$ $M = \langle W, \mu : S \to [0,1] \times \{\text{ON}, \text{OFF}\}\rangle$ with a ternary aggregation A, we will present the process of constructing a fuzzy graph (with no high-order arrow) from M based on A. In addition, we will relate the generated fuzzy graph to a finite set of arrows associated to zero-order arrows in M.

4.1. Induced fuzzy graphs from RFSGs

Consider a $RFSG$ M and an aggregation function A.

Definition 4.1. Given a $RFSG$ M with a ternary aggregation function A, let be the family of admissible fuzzy subsets of S, Ω, which is the least set s.t.,

$$\begin{cases} \mu \in \Omega \\ \overline{\mu}^A_{a_i^0} \in \Omega, \text{ whenever } \overline{\mu} \in \Omega \text{ and } a_i^0 \in S^{0\,*}_{\overline{\mu}} \end{cases}$$

Consider $\tilde{W} = \{(w, \mu) \in W \times \Omega\}$ and $\tilde{R} : \tilde{W} \times \tilde{W} \to [0,1]$ s.t.

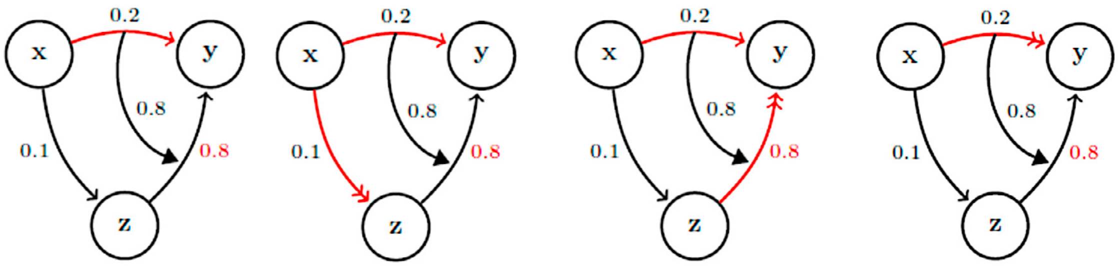
$$\tilde{R}\Big((w, \mu), (w', \mu')\Big) = \begin{cases} \mu_1[ww'], \text{ if } \mu' = \mu^A_{[ww']} \\ 0, \text{ otherwise.} \end{cases}$$

The fuzzy graph $\tilde{M} = \langle \tilde{W}, \tilde{R}\rangle$ is called the **fuzzy graph induced based on A**.

Arrows that have a zero fuzzy value are not represented in the induced graph since they represent paths over the $RFSG$ that cannot be traversed. In the next examples, this situation will be exposed.

Example 4.1. Consider the $RFSG$ M in Fig. 9(a). We have $W = \{x, y, z\}$ and considering the aggregation $A(x, y, z) = y$, we have $\Omega = \left\{\mu, \mu^A_{[xy]}\right\}$. Indeed, $\mu^A_{[xz]} = \mu$ and $\mu^A_{[xy][xz]} = \mu^A_{[xy][zy]} = \mu^A_{[xy][xy]} = \mu^A_{[xy]}$ (see Fig. 9(b) and Fig. 10).

Denote $\mu^A_{[xy]} = \overline{\mu}$ and define $\tilde{W} = \{(x, \mu), (y, \mu), (z, \mu), (x, \overline{\mu}), (y, \overline{\mu}), (z, \overline{\mu})\}$. The fuzzy graph induced based on A is presented in Fig. 11.



(a) $\mu^A_{[xy]}$ (b) $\mu^A_{[xy][xz]}$ (c) $\mu^A_{[xy][zy]}$ (d) $\mu^A_{[xy][xy]}$

Fig. 10. $M^A_{[xy]}$, $M^A_{[xy][xz]}$, $M^A_{[xy][zy]}$ and $M^A_{[xy][xy]}$.

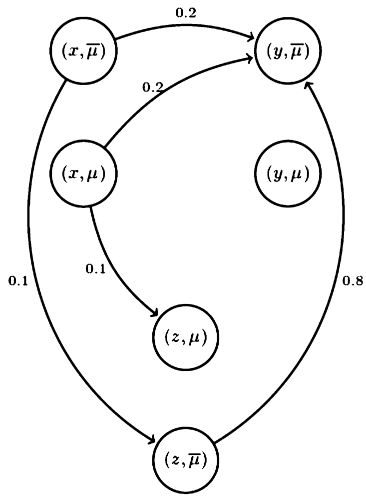
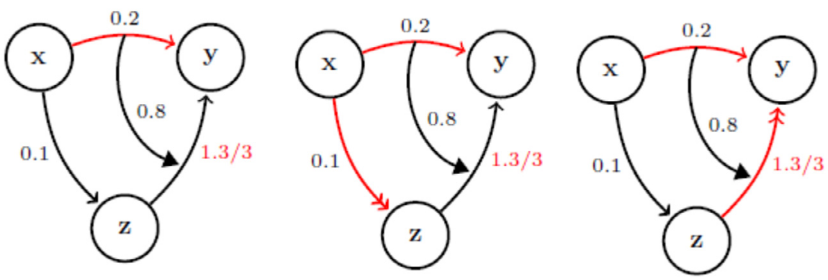


Fig. 11. Fuzzy graph induced based on A.



(a) $\mu^A_{[xy]}$ (b) $\mu^A_{[xy][xz]}$ (c) $\mu^A_{[xy][zy]}$

Fig. 12. $M^A_{[xy]}$, $M^A_{[xy][xz]}$ and $M^A_{[xy][zy]}$.

Example 4.2. Consider the same $RFSG$ M in Fig.8(a) with the aggregation $A(x, y, z) = (x + y + z)/3$. In this case, we have $\Omega = \left\{\mu, \mu^A_{[xy]}, \mu^A_{[xy][xy]}, \mu^A_{[xy][xy][xy]}, \cdots, \mu^A_{[xy]^n};\ n \in \mathbb{N}\right\}$[2] with $\mu^A_{[xz]} = \mu$ and $\mu^A_{[xy]^n[xz]} = \mu^A_{[xy]^n[zy]} = \mu^A_{[xy]^n}$ for $n \in \mathbb{N}$, as can be seen Fig. 9, Fig. 12 and Fig. 13.

Fig. 14 shows the **fuzzy graph induced based on A**. We will denote $\overline{\mu} = \mu^A_{[xy]}$, $\overline{\overline{\mu}} = \mu^A_{[xy][xy]}$ and so on.

From the examples above, we can see that the induced fuzzy graph remains finite for the second projection whereas becomes infinite for the arithmetic mean. This fact illustrates that, the aggregation properties influence the type of induced

[2] If the arrow $[xy]$ is crossed n times, repeatedly, we denote $[xy]^n$.

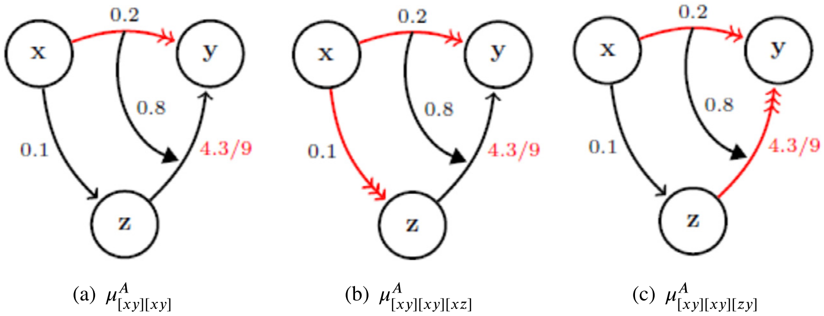


Fig. 13. $M^A_{[xy][xy]}$, $M^A_{[xy][xy][xz]}$ and $M^A_{[xy][xy][zy]}$.

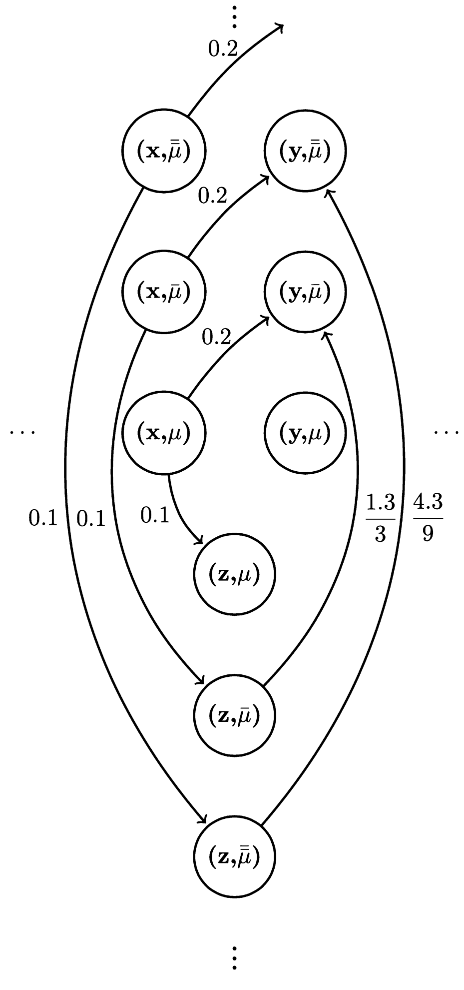


Fig. 14. $\tilde{M}$.

graph resulting and, for some cases, infinite fuzzy graphs can be represented by finite $RFSG$. The process of reducing infinite fuzzy graph to a finite reactive fuzzy graph ($RFSG$ or FSG) is expected to be studied in future works.

4.2. Induced fuzzy graph like a generated algebra

The next theorem presents the process of setting up an induced fuzzy graph from a finite set X. This process is important since it points to a recursive process for building fuzzy graphs (finite or infinite) from a finite set of arrows.

Theorem 4.1. *Given a RFSG $M = \langle W, \mu\rangle$, a ternary aggregation A and the fuzzy induced graph based on A, $\tilde{M} = \langle \tilde{W}, \tilde{R}\rangle$. Consider the set $X \subseteq \tilde{W} \times \tilde{W} \times [0,1]$ s.t.*

$$X = \left\{ \Big((w,\mu), (w',\mu), \tilde{R}\Big((w,\mu),(w',\mu)\Big)\Big); [ww'] \in S^0 \right\}.$$

and the building rule $X_0 = X$ and

$$X_{j+1} = X_j \cup \left\{ \left\{ \bigcup_{a\in S_\to} f_a\big(X_j\big) \right\} - \left\{ \Big((w,s),(w',s'),d\Big) \in \left\{ \bigcup_{a\in S_\to} f_a\big(X_j\big)\right\};\ \big(s'\big)_2[ww'] = \text{OFF} \right\} \right\}$$

for $f_a : \tilde{W} \times \tilde{W} \times [0,1] \to \tilde{W} \times \tilde{W} \times [0,1]$ with $a \in S_\to$ s.t.

$$f_a\Big((w,\phi),(w',\phi'),d\Big) = \begin{cases} \Big((w,\phi'),(w',\phi'^A_a),\big(\phi'^A_a\big)_1[ww']\Big), & \text{if } \phi'^A_a \in \Omega \text{ and } [ww'] = a \\ \Big((w,\phi'^A_a),(w',\phi'^A_a),\big(\phi'^A_a\big)_1[ww']\Big), & \text{if } \phi'^A_a \in \Omega \text{ and } [ww'] \neq a. \\ \Big((w,\phi),(w',\phi'),d\Big), & \text{if } \phi'^A_a \notin \Omega. \end{cases}$$

Then $\tilde{M} = \langle X\rangle = \bigcup_{j\in\mathbb{N}} X_j$.

Proof. Indeed,

i) $\langle X\rangle \subseteq \tilde{M}$: We prove this result by induction.
Note that, by definition, $X \in \tilde{M}$. Consider $f_\alpha \in \mathcal{F} = \Big\{ f_\alpha : \tilde{W} \times \tilde{W} \times [0,1] \to \tilde{W} \times \tilde{W} \times [0,1];\ \alpha \in S_\to \Big\}$ and $\Big((w,s),(w',s'),\tilde{R}\big((w,s),(w',s')\big)\Big) \in \tilde{M}$, then:
If $\alpha = [ww']$ and $s'^A_\alpha \in \Omega$:

$$\begin{aligned} f_\alpha\Big((w,s),(w',s'),\tilde{R}\big((w,s),(w',s')\big)\Big) &= \Big((w,s'),(w',s'^A_\alpha),\big(s'^A_\alpha\big)_1[ww']\Big) \\ &= \Big((w,s'),(w',s'^A_\alpha),\tilde{R}\big((w,s'^A_\alpha),(w',s'^A_{\alpha[ww']})\big)\Big) \in \tilde{M} \end{aligned}$$

If $\alpha \neq [ww']$ and $s'^A_\alpha \in \Omega$:

$$\begin{aligned} f_\alpha\Big((w,s),(w',s'),\tilde{R}\big((w,s),(w',s')\big)\Big) &= \Big((w,s'^A_\alpha),(w',s'^A_\alpha),\big(s'^A_\alpha\big)_1[ww']\Big) \\ &= \Big((w,s'^A_\alpha),(w',s'^A_\alpha),\tilde{R}\big((w,s'^A_\alpha),(w',s'^A_{\alpha[ww']})\big)\Big) \in \tilde{M} \end{aligned}$$

If $s'^A_\alpha \notin \Omega$:

$$f_\alpha\Big((w,s),(w',s'),\tilde{R}\big((w,s),(w',s')\big)\Big) = \Big((w,s),(w',s'),\tilde{R}\big((w,s),(w',s')\big)\Big) \in \tilde{M}$$

Therefore, $\tilde{M}$ is closed in relation to the functions in $\mathcal{F}$. Supposing $X_j \subseteq \tilde{M}$, for $j \in \mathbb{N}$. Then, $X_{j+1} = X_j \cup \left\{ \left\{ \bigcup_{a\in S_\to} f_a\big(X_j\big) \right\} - \left\{ \Big((w,s),(w',s'),d\Big) \in \left\{ \bigcup_{a\in S_\to} f_a\big(X_j\big)\right\};\ \big(s'\big)_2[ww'] = \text{OFF} \right\} \right\} \subseteq \tilde{M}$. Therefore, $\langle X\rangle \subseteq \tilde{M}$.

ii) $\tilde{M} \subseteq \langle X\rangle$:
Indeed, consider $\alpha, \beta \in S^*$ s.t. $\alpha \in S_\to$ and $\Big((w,\mu^A_\alpha),(w',\mu^A_\beta),\big(\mu^A_\beta\big)_1[ww']\Big) \in \tilde{M}$.
If $\mu^A_\beta = \mu^A_{\alpha[ww']} \neq \mu^A_\alpha$, there are $j \in \mathbb{N}$ and $\Big((w,\mu),(w',\mu^A_\alpha),\big(\mu^A_\alpha\big)_1[ww']\Big) \in X_{j-1}$ s.t.

$$\begin{aligned} f_\alpha\Big((w,\mu),(w',\mu^A_\alpha),\big(\mu^A_\alpha\big)_1[ww']\Big) &= \Big((w,\mu^A_\alpha),(w',\mu^A_{\alpha[ww']}),\big(\mu^A_{\alpha[ww']}\big)_1[ww']\Big) \in X_j \\ &= \Big((w,\mu^A_\alpha),(w',\mu^A_\beta),\big(\mu^A_\beta\big)_1[ww']\Big) \in X_j \end{aligned}$$

If $\mu^A_\beta = \mu^A_{\alpha[ww']} = \mu^A_\alpha$, there are $j \in \mathbb{N}$ and $\Big((w,\mu^A_\alpha),(w',\mu^A_\alpha),\big(\mu^A_\alpha\big)_1[ww']\Big) \in X_{j-1}$ s.t.

$$\begin{aligned} f_\alpha\Big((w,\mu^A_\alpha),(w',\mu^A_\alpha),\big(\mu^A_\alpha\big)_1[ww']\Big) &= \Big((w,\mu^A_{\alpha[ww']}),(w',\mu^A_{\alpha[ww']}),\big(\mu^A_{\alpha[ww']}\big)_1[ww']\Big) \in X_j \\ &= \Big((w,\mu^A_\alpha),(w',\mu^A_\beta),\big(\mu^A_\beta\big)_1[ww']\Big) \in X_j \end{aligned}$$

Therefore, if $\Big((w,\mu^A_\alpha),(w',\mu^A_\beta),(\mu^A_\beta)_1[ww']\Big)\in\tilde{M}$, there is $j\in\mathbb{N}$ s.t. $\Big((w,\mu^A_\alpha),(w',\mu^A_\beta),(\mu^A_\beta)_1[ww']\Big)\in X_j\subseteq\langle X\rangle$.

Due the items (i) and (ii), $\tilde{M}=\langle X\rangle$.

In the following, we will present two examples of how a fuzzy induced graph (finite and infinite) can be written as algebra generated by a finite set of arrows.

Example 4.3. Given the $RFSG$ in Fig. 18(a) and its induced fuzzy graph in Fig. 11 (Example 4.1). In this case, we have $S_\rightarrow=\{[xy]\}$ and

$$\begin{aligned}X&=\Big\{\Big((x,\mu),(y,\mu),\tilde{R}\big((x,\mu),(y,\mu)\big)\Big),\Big((x,\mu),(z,\mu),\tilde{R}\big((x,\mu),(z,\mu)\big)\Big),\Big((z,\mu),(y,\mu),\tilde{R}\big((z,\mu),(y,\mu)\big)\Big)\Big\}\\&=\Big\{\Big((x,\mu),(y,\mu),0\Big),\Big((x,\mu),(z,\mu),0.1\Big),\Big((z,\mu),(y,\mu),0\Big)\Big\}.\end{aligned}$$

Let be

$$f_{[xy]}\Big((w,\phi),(w',\phi'),d\Big)=\begin{cases}\Big((w,\phi'),(w',\phi'^A_{[xy]}),(\phi'^A_{[xy]})_1[ww']\Big), \text{ if } \phi'^A_{[xy]}\in\Omega \text{ and } [ww']=[xy]\\ \Big((w,\phi'^A_{[xy]}),(w',\phi'^A_{[xy]}),(\phi'^A_{[xy]})_1[ww']\Big), \text{ if } \phi'^A_{[xy]}\in\Omega \text{ and } [ww']\neq[xy].\\ \Big((w,\phi),(w',\phi'),d\Big), \text{ if } \phi'^A_{[xy]}\notin\Omega.\end{cases}$$

and $X_0=X$, we calculate:

- $f_{[xy]}\Big((x,\mu),(y,\mu),0\Big)=\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big)$ due $\mu^A_{[xy]}\in\Omega$ and $[xy]=[xy]$;
- $f_{[xy]}\Big((x,\mu),(z,\mu),0.1\Big)=\Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big)$ due $\mu^A_{[xy]}\in\Omega$ and $[xz]\neq[xy]$;
- $f_{[xy]}\Big((z,\mu),(y,\mu),0\Big)=\Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.8\Big)$ due $\mu^A_{[xy]}\in\Omega$ and $[zy]\neq[xy]$.

Observe that $f_{[xy]}\Big(X_0\Big)=\Big\{\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big),\Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big),\Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.8\Big)\Big\}$ and $\Big\{\Big((w,s),(w',s'),d\Big)\in f_{[xy]}\Big(X_0\Big);\ (s')_2[ww']=\textsc{off}\Big\}=\emptyset$. Then,

$$\begin{aligned}X_1&=X_0\cup f_{[xy]}\Big(X_0\Big)\\&=\Big\{\Big((x,\mu),(y,\mu),0\Big),\Big((x,\mu),(z,\mu),0.1\Big),\Big((z,\mu),(y,\mu),0\Big),\\&\quad\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big),\Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big),\Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.8\Big)\Big\}\end{aligned}$$

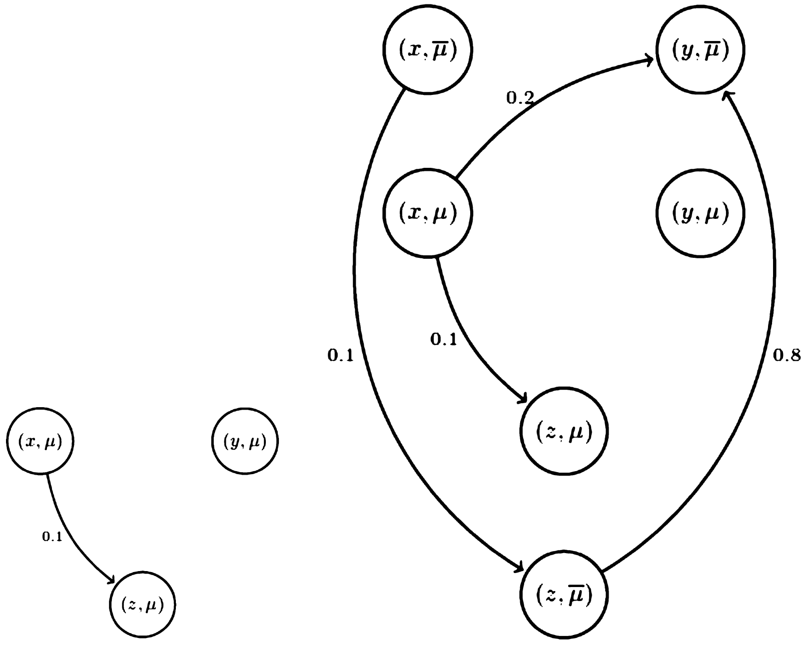
Fig. 15(a) and Fig. 15(b) show the sets X_0 and X_1. Continuing the process, we will calculate X_2 showing the images of the arrows in $f_{[xy]}(X_0)$:

- $f_{[xy]}\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big)=\Big((x,\mu^A_{[xy]}),(y,\mu^A_{[xy][xy]}),0.2\Big)=\Big((x,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.2\Big)$ due $\mu^A_{[xy][xy]}=\mu^A_{[xy]}\in\Omega$ and $[xy]=[xy]$;
- $f_{[xy]}\Big((x,\mu),(z,\mu),0.1\Big)=\Big((x,\mu^A_{[xy][xy]}),(z,\mu^A_{[xy][xy]}),0.1\Big)=\Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big)$ due $\mu^A_{[xy][xy]}=\mu^A_{[xy]}\in\Omega$ and $[xz]\neq[xy]$;
- $f_{[xy]}\Big((z,\mu),(y,\mu),0\Big)=\Big((z,\mu^A_{[xy][xy]}),(y,\mu^A_{[xy][xy]}),0.8\Big)=\Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.8\Big)$ due $\mu^A_{[xy][xy]}=\mu^A_{[xy]}\in\Omega$ and $[zy]\neq[xy]$.

Then, $f_{[xy]}\Big(X_1\Big)=f_{[xy]}\Big(X_0\Big)\cup\Big\{\Big((x,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.2\Big),\Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big),\Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.8\Big)\Big\}$ and $\Big\{\Big((w,s),(w',s'),d\Big)\in f_{[xy]}\Big(X_1\Big);\ (s')_2[ww']=\textsc{off}\Big\}=\emptyset$.

Follow that,

$$\begin{aligned}X_2&=X_1\cup f_{[xy]}\Big(X_1\Big)\\&=\Big\{\Big((x,\mu),(y,\mu),0\Big),\Big((x,\mu),(z,\mu),0.1\Big),\Big((z,\mu),(y,\mu),0\Big),\\&\quad\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big),\Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big),\Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.8\Big),\Big((x,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.2\Big)\Big\}\end{aligned}$$



(a) X_0 (b) X_1

Fig. 15. Sets of arrows that make up the induced fuzzy graph $\tilde{M}$.

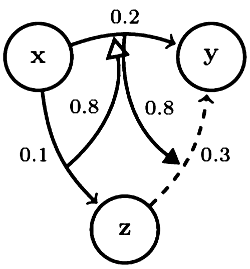


Fig. 16. $RFSG$ M.

Due to the aggregation used, when crossing n times ($n \in \mathbb{N}$) the arrow $[xy]$, the update application $\mu^A_{[xy]^n}$ will be overlaid on the set Ω by the application $\mu^A_{[xy]}$. Thus, the sets $X_3, X_4, ..., X_n = X_2$ and the induced fuzzy graph based on A from M will be generated, like an algebra, by the finite set X. Fig. 11 shows the set X_2.

Example 4.4. Given the $RFSG$ M in Fig. 16. The induced fuzzy graph based in product can be viewed in Fig. 17.

Consider the base set $X = \Big\{\Big((x,\mu),(y,\mu),0\Big),\Big((x,\mu),(z,\mu),0\Big),\Big((z,\mu),(y,\mu),0\Big)\Big\}$ and the set $S_\to = \Big\{[xy],[xz]\Big\}$. We get $X_0 = X$ (see Fig. 18 (a)) and calculating:

- $f_{[xy]}\Big((x,\mu),(y,\mu),0\Big) = \Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big)$;
- $f_{[xy]}\Big((x,\mu),(z,\mu),0\Big) = \Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big)$;
- $f_{[xy]}\Big((z,\mu),(y,\mu),0\Big) = \Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.048\Big)$;
- $f_{[xz]}\Big((x,\mu),(y,\mu),0\Big) = \Big((x,\mu^A_{[xz]}),(y,\mu^A_{[xz]}),0.016\Big)$;
- $f_{[xz]}\Big((x,\mu),(z,\mu),0\Big) = \Big((x,\mu),(z,\mu^A_{[xz]}),0.1\Big)$;
- $f_{[xz]}\Big((z,\mu),(y,\mu),0\Big) = \Big((z,\mu^A_{[xz]}),(y,\mu^A_{[xz]}),0.3\Big)$.

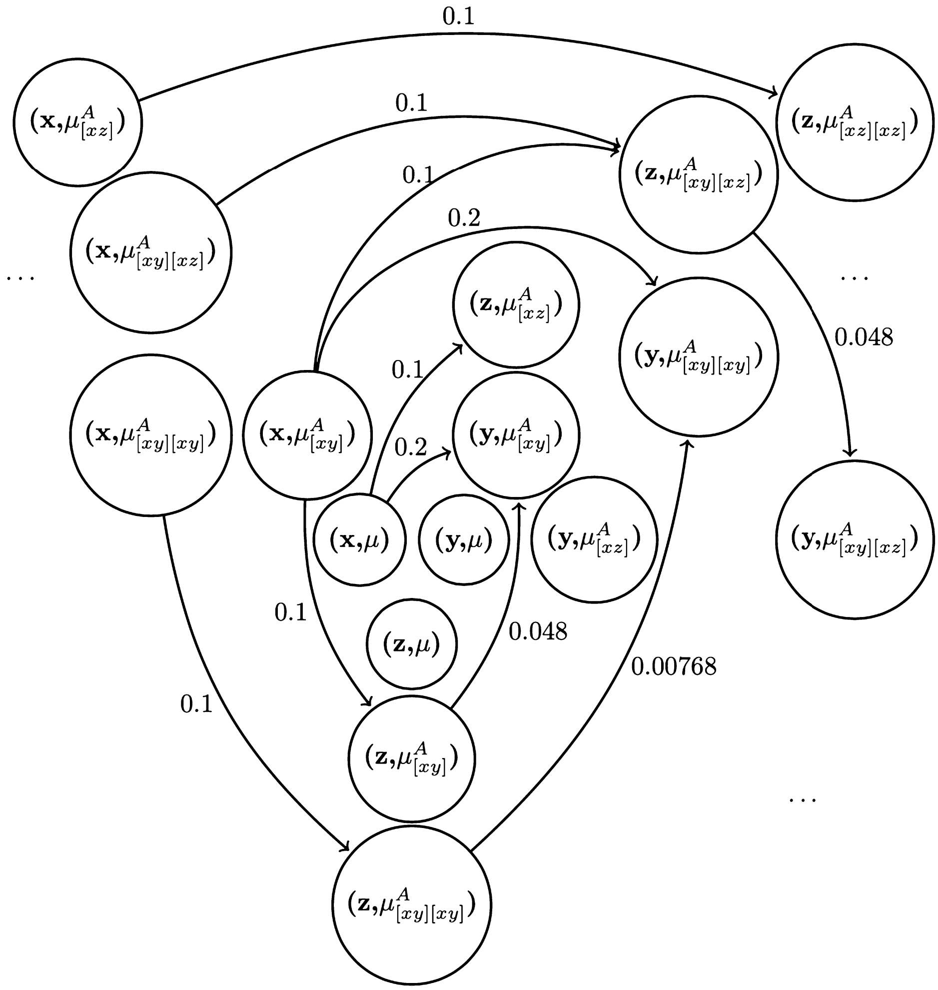


Fig. 17. Fuzzy induced graph $\tilde{M}$.

We get

$$\left\{f_{[xy]}\big(X_0\big), f_{[xz]}\big(X_0\big)\right\} = \Big\{\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big), \Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big), \Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.048\Big), \\ \Big((x,\mu^A_{[xz]}),(y,\mu^A_{[xz]}),0.016\Big), \Big((x,\mu),(z,\mu^A_{[xz]}),0.1\Big), \Big((z,\mu^A_{[xz]}),(y,\mu^A_{[xz]}),0.3\Big), \Big\}$$

and

$$\Big\{\Big((w,s),(w',s'),d\Big) \in \left\{f_{[xy]}\big(X_0\big), f_{[xz]}\big(X_0\big)\right\}; \big(s'\big)_2[ww'] = \text{OFF}\Big\} \\ = \Big\{\Big((x,\mu^A_{[xz]}),(y,\mu^A_{[xz]}),0.016\Big), \Big((z,\mu^A_{[xz]}),(y,\mu^A_{[xz]}),0.3\Big)\Big\}.$$

Therefore, as can be seen in Fig. 18 (b),

$$X_1 = X_0 \cup \Big\{\left\{f_{[xy]}(X_0), f_{[xz]}(X_0)\right\} - \Big\{\Big((x,\mu^A_{[xz]}),(y,\mu^A_{[xz]}),0.016\Big)\Big\}\Big\} \\ = X_0 \cup \Big\{\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big), \Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big), \Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.048\Big), \\ \Big((x,\mu),(z,\mu^A_{[xz]}),0.1\Big)\Big\}$$

To calculate X_2, we have:

- $f_{[xy]}\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big) = \Big((x,\mu^A_{[xy]}),(y,\mu^A_{[xy][xy]}),0.2\Big)$;

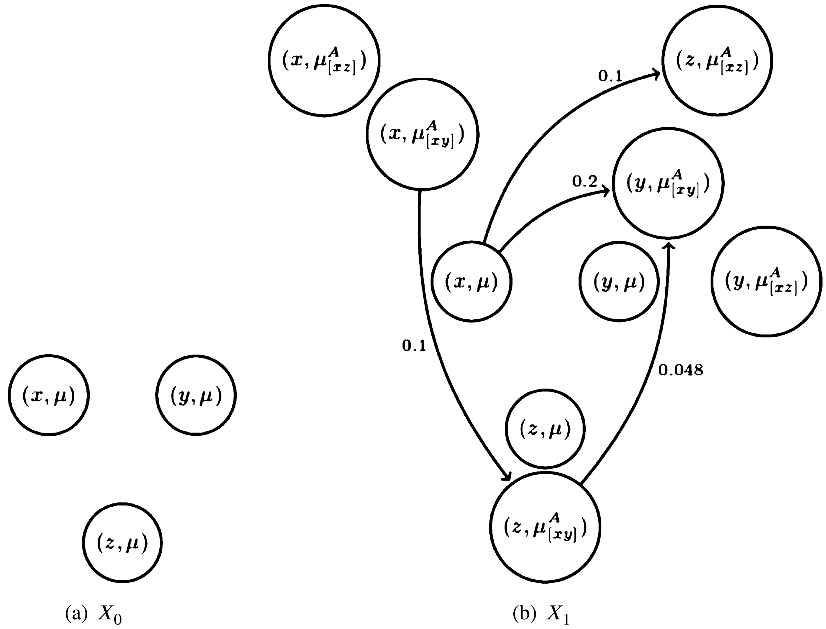


Fig. 18. Sets of arrows that make up the induced fuzzy graph $\tilde{M}$.

- $f_{[xy]}\Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big)=\Big((x,\mu^A_{[xy][xy]}),(z,\mu^A_{[xy][xy]}),0.1\Big)$;
- $f_{[xy]}\Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.048\Big)=\Big((z,\mu^A_{[xy][xy]}),(y,\mu^A_{[xy][xy]}),0,00768\Big)$;
- $f_{[xy]}\Big((x,\mu),(z,\mu^A_{[xz]}),0.1\Big)=\Big((x,\mu),(z,\mu^A_{[xz]}),0.1\Big)$ due $\mu^A_{[xz][xy]}\notin\Omega$.

and

- $f_{[xz]}\Big((x,\mu),(y,\mu^A_{[xy]}),0.2\Big)=\Big((x,\mu^A_{[xy][xz]}),(y,\mu^A_{[xy][xz]}),0.016\Big)$;
- $f_{[xz]}\Big((x,\mu^A_{[xy]}),(z,\mu^A_{[xy]}),0.1\Big)=\Big((x,\mu^A_{[xy][xz]}),(z,\mu^A_{[xy][xz]}),0.1\Big)$;
- $f_{[xz]}\Big((z,\mu^A_{[xy]}),(y,\mu^A_{[xy]}),0.048\Big)=\Big((z,\mu^A_{[xy][xz]}),(y,\mu^A_{[xy][xz]}),0.048\Big)$;
- $f_{[xz]}\Big((x,\mu),(z,\mu^A_{[xz]}),0.1\Big)=\Big((x,\mu^A_{[xz][xz]}),(z,\mu^A_{[xz][xz]}),0.1\Big)$.

We get

$$\begin{aligned}\big\{f_{[xy]}(X_1),f_{[xz]}(X_1)\big\}=\Big\{&\Big((x,\mu^A_{[xy]}),(y,\mu^A_{[xy][xy]}),0.2\Big),\Big((x,\mu^A_{[xy][xy]}),(z,\mu^A_{[xy][xy]}),0.1\Big),\\&\Big((z,\mu^A_{[xy][xy]}),(y,\mu^A_{[xy][xy]}),0,00768\Big),\Big((x,\mu^A_{[xy][xz]}),(y,\mu^A_{[xy][xz]}),0.016\Big),\\&\Big((x,\mu^A_{[xy][xz]}),(z,\mu^A_{[xy][xz]}),0.1\Big),\Big((z,\mu^A_{[xy][xz]}),(y,\mu^A_{[xy][xz]}),0.048\Big),\\&\Big((x,\mu^A_{[xz][xz]}),(z,\mu^A_{[xz][xz]}),0.1\Big)\Big\}\end{aligned}$$

and $\Big\{\Big((w,s),(w',s'),d\Big)\in\big\{f_{[xy]}(X_1),f_{[xz]}(X_1)\big\};(s')_2[ww']=\text{OFF}\Big\}=\Big\{\Big((x,\mu^A_{[xy][xz]}),(y,\mu^A_{[xy][xz]}),0.016\Big)\Big\}$.

Therefore, as can be seen in Fig. 16,

$$\begin{aligned}X_2=X_1\,\cup\,\Big\{&\Big((x,\mu^A_{[xy]}),(y,\mu^A_{[xy][xy]}),0.2\Big),\Big((x,\mu^A_{[xy][xy]}),(z,\mu^A_{[xy][xy]}),0.1\Big),\\&\Big((z,\mu^A_{[xy][xy]}),(y,\mu^A_{[xy][xy]}),0,00768\Big),\Big((x,\mu^A_{[xy][xz]}),(z,\mu^A_{[xy][xz]}),0.1\Big),\\&\Big((z,\mu^A_{[xy][xz]}),(y,\mu^A_{[xy][xz]}),0.048\Big),\Big((x,\mu^A_{[xz][xz]}),(z,\mu^A_{[xz][xz]}),0.1\Big)\Big\}.\end{aligned}$$

The process goes on to determine $X_n, n\geq 3$. The graph $\tilde{M}$ is built from these sets and is an infinite graph.

5. A logic for $RFSG$s

In order to verify a system described by a $RFSG$, we provide a formal language and a fuzzy semantics. Also in this section, we will present the definition of simulation and bisimulation for $RFSG$s. In what follows, for any $w \in W$, we use the set $S^{0*}[w] = \{w' \in W; [ww'] \in S^0\}$.

5.1. Syntax and semantics

In [7] was present a formal logic for $RFSG$s which enables the verification of properties. This section expose this logic with more details and introduce new concepts.

Definition 5.1 *(Syntax [7]).* Consider *AtomProp* a set of symbols (called atomic propositions) and $p \in AtomProp$. The set of formulas is generated by the following grammar: $\varphi ::= p \mid true \mid false \mid (\neg\varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid (\varphi \leftrightarrow \varphi) \mid (S_{Next}(\varphi)) \mid (A_{Next}(\varphi))$.

Given the formulas φ and ψ, we classically interpret:

$(\neg\varphi)$: φ is not true;
$(\varphi \wedge \psi)$: φ and ψ are true;
$(\varphi \vee \psi)$: φ or ψ is true;
$(\varphi \to \psi)$: If φ is true, then ψ is true;
$(\varphi \leftrightarrow \psi)$: φ is true if and only if ψ is true;
$(S_{Next}(\varphi))$: φ is true in some next state;
$(A_{Next}(\varphi))$: φ is true in all next states.

A formula that only contains the operators $\wedge, \vee$ and $S_{Next}(\varphi)$ is called **positive formula**.

Definition 5.2. A **model [7]** over the set *AtomProp* is a pair $\mathscr{M} = (M, V_M)$, s.t. $M = \langle W, \mu\rangle$ is a $RFSG$ and $V_M : W \times AtomProp \to [0,1]$ is a function called **fuzzy valuation**.

Definition 5.3. Given a model $\mathscr{M} = (M, V_M)$ and N a subgraph of M, the structure $\mathscr{N} = (N, V_N)$ is a **submodel** of $\mathscr{M}$ whenever $V_N(w,p) \leq V_M(w,p)$ for all $w \in W$ and $p \in AtomProp$.

Definition 5.4 *(Semantics [7]).* Consider $\mathscr{M} = (M, V_M)$ a model, A the aggregation function associated with M, $\mathcal{F} = \langle [0,1], T, S, N, I, B, 0, 1\rangle$ a fuzzy semantics and $w \in W$ a state. The notation, $[\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!]$ represents the **grade of uncertainty of a given formula φ, at state w, taking into account $\mathscr{M}, \mathcal{F}$ and A**. The grade of uncertainty of $[\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!]$ is defined in the following way:

- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} p]\!] = V_M(w,p)$, for $p \in$ AtomProp.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} \text{true}]\!] = 1$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} \text{false}]\!] = 0$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \wedge \psi)]\!] = \mathbf{T}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!])$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \vee \psi)]\!] = \mathbf{S}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!])$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \to \psi)]\!] = \mathbf{I}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!])$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \leftrightarrow \psi)]\!] = \mathbf{B}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!])$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} \neg\varphi)]\!] = \mathbf{N}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!])$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} A_{Next}(\varphi)]\!] = \underset{w' \in S^{0*}[w]}{\mathbf{T}} \Big(\mathbf{I}\Big(\mu([ww']), [\![\mathscr{M}^A_{[ww']}, w' \models^A_{\mathcal{F}} \varphi]\!]\Big)\Big)$ since $\mathscr{M}^A_{[ww']}$ means $\Big(M^A_{[ww']}, V_M\Big)$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} S_{Next}(\varphi)]\!] = \underset{w' \in S^{0*}[w]}{\mathbf{S}} \Big(\mathbf{T}\Big(\mu([ww']), [\![\mathscr{M}^A_{[ww']}, w' \models^A_{\mathcal{F}} \varphi]\!]\Big)\Big)$.

The uncertainty degree that “$S_{Next}(\varphi)$” is true at the state w is computed by using the uncertainty degree that φ is true at *some* state with active relationship to w. On the other hand, the uncertainty degree that “$A_{Next}(\varphi)$” is true at state w is computed by using the uncertainty degree that φ is true at *every* state with active relationship to w. The expression: $[\![\mathscr{M}^A_{[w,w']}, w' \models \varphi]\!]$, in this case, represents the uncertainty degree of the statement: “*φ is true*” at state w' after the active zero-order arrow $a^0_i = [w, w']$ has been crossed and the $RFSG$ M has been updated to $M^A_{a^0_i}$.

Remark 5.1. According to Notation 1, the application f in the definition of $A_{Next}(\varphi)$ is a fuzzy implication **I**. Similarly, in the definition of $S_{Next}(\varphi)$, the application f is a t-norm **T**.

Table 2
Truth values of propositions on each state.

	x	y	u	v
H	0.2	0.8	0.3	0.01
L	0.1	0.9	0.15	0.2

Example 5.1. Consider Fig. 1(b) and take the atomic propositions: *High risk of contagion* and *Low risk of contagion*, according to the Table 2.

What is the uncertainty degree at state x for the proposition: *"In some next state we have a low risk of contagion with a next state which has a higher risk of contagion?"* The assertion can be expressed as: $S_{Next}(L \wedge S_{Next}(H))$.

Assuming the *arithmetic mean* as the unique aggregation function, the *Gödel Semantic* $\mathcal{F}_G$ and $\varphi = L \wedge S_{Next}(H)$,

$$[\![\mathscr{M}, x \models^A_{\mathcal{F}_G} S_{Next}(\varphi)]\!] \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M\big(0.3, [\![\mathscr{M}^A_{[xu]}, u \models^A_{\mathcal{F}_G} \varphi]\!]\big), \mathbf{T}_M\big(0.4, [\![\mathscr{M}^A_{[xy]}, y \models^A_{\mathcal{F}_G} \varphi]\!]\big)\Big)$$
$$\stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M(0.3, 0.15), \mathbf{T}_M(0.4, 0.01)\Big) = 0.15$$

Since,

a) $[\![\mathscr{M}^A_{[xu]}, u \models^A_{\mathcal{F}_G} \varphi]\!] \stackrel{\text{def}}{=} \mathbf{T}_M\Big([\![\mathscr{M}^A_{[xu]}, u \models^A_{\mathcal{F}_G} S_{Next}(H)]\!], [\![\mathscr{M}^A_{[xu]}, u \models^A_{\mathcal{F}_G} L]\!]\Big) = \mathbf{T}_M(0.6, 0.15) = 0.15$ due to $[\![\mathscr{M}^A_{[xu]}, u \models^A_{\mathcal{F}_G} S_{Next}(H)]\!] \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M\big(\mu^A_{[xu]}([uy]), [\![\mathscr{M}^A_{[xu][uy]}, y \models^A_{\mathcal{F}_G} H]\!]\big)\Big) = \mathbf{S}_M\Big(\mathbf{T}_M\big(0.6, 0.8\big)\Big) = 0.6$;

b) $[\![\mathscr{M}^A_{[xy]}, y \models^A_{\mathcal{F}_G} \varphi]\!] \stackrel{\text{def}}{=} \mathbf{T}_M\Big([\![\mathscr{M}^A_{[xy]}, y \models^A_{\mathcal{F}_G} S_{Next}(H)]\!], [\![\mathscr{M}^A_{[xy]}, y \models^A_{\mathcal{F}_G} L]\!]\Big) = \mathbf{T}_M(0.01, 0.9) = 0.01.$ due to $[\![\mathscr{M}^A_{[xy]}, y \models^A_{\mathcal{F}_G} S_{Next}(H)]\!] \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M\big(\mu^{[xy]}_A([yv]), [\![\mathscr{M}^A_{[xy][yv]}, v \models^A_{\mathcal{F}_G} H]\!]\big)\Big) = \mathbf{S}_M\Big(\mathbf{T}_M\big(0.02, 0.01\big)\Big) = 0.01$

In this case, in order to calculate the uncertainty degree at state v for the same proposition, we should note that the state v has only one next state y (the inactive arrow $[vu]$ is not considered). Therefore,

$$[\![\mathscr{M}, v \models^A_{\mathcal{F}_G} S_{Next}(\varphi)]\!] \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M\big(0.03, [\![\mathscr{M}^A_{[vy]}, y \models^A_{\mathcal{F}_G} \varphi]\!]\big)\Big) = 0.01.$$

Since $[\![\mathscr{M}^A_{[vy]}, y \models^A_{\mathcal{F}_G} \varphi]\!] \stackrel{\text{def}}{=} \mathbf{T}_M\Big([\![\mathscr{M}^A_{[vy]}, y \models^A_{\mathcal{F}_G} S_{Next}(H)]\!], [\![\mathscr{M}^A_{[vy]}, y \models^A_{\mathcal{F}_G} L]\!]\Big) = \mathbf{T}_M(0.01, 0.9) = 0.01$ due to $[\![\mathscr{M}^A_{[vy]}, y \models^A_{\mathcal{F}_G} S_{Next}(H)]\!] \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M\big(\mu^A_{[vy]}([yv]), [\![\mathscr{M}^A_{[vy][yv]}, v \models^A_{\mathcal{F}_G} H]\!]\big)\Big) = \mathbf{S}_M\Big(\mathbf{T}_M\big(0.02, 0.01\big)\Big) = 0.01.$

Proposition 5.1. *Consider $\mathscr{N} = (N, V_N)$ a submodel of $\mathscr{M} = (M, V_M)$, then*

$$[\![\mathscr{N}, w \models^A_{\mathcal{F}} \psi]\!] \leq [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!]$$

for all positive formula ψ.

Proof. We prove this result by induction over the structure of formulas.

- It holds for atomic propositions by definition and trivially for *true* and *false*.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \vee \psi)]\!] = \mathbf{T}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!]) \geq \mathbf{T}([\![\mathscr{N}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{N}, w \models^A_{\mathcal{F}} \psi]\!]) = [\![\mathscr{N}, w \models^A_{\mathcal{F}} (\varphi \vee \psi)]\!]$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \wedge \psi)]\!] = \mathbf{S}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!]) \geq \mathbf{S}([\![\mathscr{N}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{N}, w \models^A_{\mathcal{F}} \psi]\!]) = [\![\mathscr{N}, w \models^A_{\mathcal{F}} (\varphi \wedge \psi)]\!]$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} S_{Next}(\varphi)]\!] = \mathop{\mathbf{S}}_{w' \in S^{0*}[w]}\big(\mathbf{T}\big(\mu_{M1}([ww']), [\![\mathscr{M}^A_{[ww']}, w' \models^A_{\mathcal{F}} \varphi]\!]\big)\big) \geq \mathop{\mathbf{S}}_{w' \in S^{0*}[w]}\big(\mathbf{T}\big(\mu_{N1}([ww']), [\![\mathscr{N}^A_{[ww']}, w' \models^A_{\mathcal{F}} \varphi]\!]\big)\big)$
  $= [\![\mathscr{N}, w \models^A_{\mathcal{F}} S_{Next}(\varphi)]\!]$

Definition 5.5. Given a $RFSG$ $M = \langle W, \mu : S \to [0,1] \times \{\text{ON}, \text{OFF}\}\rangle$ with an aggregation A and a **model** $\mathscr{M} = (M, V_M)$, the structure

$$\widetilde{\mathscr{M}} = \big(\langle \tilde{W}, \tilde{R}\rangle, \tilde{V} = V_{\langle \tilde{W}, \tilde{R}\rangle}\big)$$

s.t.

$$\tilde{V} : \tilde{W} \times \text{AtomProp} \to [0,1]$$
$$\tilde{V}\big((w,\mu),p\big) = V_M(w,p)$$

is called the **induced fuzzy model of M for** A.

Theorem 5.1. *Given a $RFSG$ $M = \langle W, \mu : S \to [0,1] \times \{\text{ON}, \text{OFF}\}\rangle$ with an aggregation A and a model $\mathscr{M} = (M, V_M)$, then*

$$[\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!] = [\![\tilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \varphi]\!]$$

Proof. We prove this result by induction over the structure of formulas.

- It holds for atomic propositions by definition and trivially for *true* and *false*.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \vee \psi)]\!] = \mathbf{T}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!]) = \mathbf{T}([\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \varphi]\!], [\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \psi]\!]) = [\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} (\varphi \vee \psi)]\!]$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \wedge \psi)]\!] = \mathbf{S}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!]) = \mathbf{S}([\![\tilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \varphi]\!], [\![\tilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \psi]\!]) = [\![\tilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} (\varphi \wedge \psi)]\!]$.
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \to \psi)]\!] = \mathbf{I}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!]) = \mathbf{I}([\![\tilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \varphi]\!], [\![\tilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \psi]\!]) = [\![\tilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} (\varphi \to \psi)]\!]$
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\varphi \leftrightarrow \psi)]\!] = \mathbf{B}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathscr{M}, w \models^A_{\mathcal{F}} \psi]\!]) = \mathbf{B}([\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \varphi]\!], [\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \psi]\!]) = [\![\tilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} (\varphi \leftrightarrow \psi)]\!]$
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} (\neg\varphi)]\!] = \mathbf{N}([\![\mathscr{M}, w \models^A_{\mathcal{F}} \varphi]\!]) = \mathbf{N}([\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} \varphi]\!]) = [\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} (\neg\varphi)]\!]$
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} S_{Next}(\varphi)]\!] = \underset{w' \in S^{0*}[w]}{\mathbf{S}} \Big(\mathbf{T}\Big(\mu_1([ww']), [\![\mathscr{M}^A_{[ww']}, w' \models^A_{\mathcal{F}} \varphi]\!]\Big)\Big) = \underset{w' \in S^{0*}[w]}{\mathbf{S}} \Big(\mathbf{T}\Big(\mu_1([ww']), [\![\widetilde{\mathscr{M}}^A_{[ww']}, (w', \mu^A_{[ww']}) \models_{\mathcal{F}} \varphi]\!]\Big)\Big) = \underset{w' \in S^{0*}[w]}{\mathbf{S}} \Big(\mathbf{T}\Big(\mu_1([ww']), [\![\widetilde{\mathscr{M}}, (w', \mu^A_{[ww']}) \models_{\mathcal{F}} \varphi]\!]\Big)\Big) = [\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} S_{Next}(\varphi)]\!]$
- $[\![\mathscr{M}, w \models^A_{\mathcal{F}} A_{Next}(\varphi)]\!] = \underset{w' \in S^{0*}[w]}{\mathbf{T}} \Big(\mathbf{I}\Big(\mu_1([ww']), [\![\mathscr{M}^A_{[ww']}, w' \models^A_{\mathcal{F}} \varphi]\!]\Big)\Big) = \underset{w' \in S^{0*}[w]}{\mathbf{T}} \Big(\mathbf{I}\Big(\mu_1([ww']), [\![\widetilde{\mathscr{M}}^A_{[ww']}, (w', \mu^A_{[ww']}) \models_{\mathcal{F}} \varphi]\!]\Big)\Big) = \underset{w' \in S^{0*}[w]}{\mathbf{T}} \Big(\mathbf{I}\Big(\mu_1([ww']), [\![\widetilde{\mathscr{M}}, (w', \mu^A_{[ww']}) \models_{\mathcal{F}} \varphi]\!]\Big)\Big) = [\![\widetilde{\mathscr{M}}, (w,\mu) \models_{\mathcal{F}} A_{Next}(\varphi)]\!]$.

5.2. Simulation and bisimulation

Based on the notion of bisimulation for FSGs present in [22], we introduce the notion of simulation and bisimulation for $RFSG$s.

Definition 5.6. [17] A **fuzzy model** over the set *AtomProp* is a pair $\mathscr{MF} = \big(\langle W, R\rangle, V_{\langle W,R\rangle}\big)$ s.t. $\langle W, R\rangle$ is a fuzzy graph and $V_{\langle W,R\rangle} : W \times AtomProp \to [0,1]$ is a fuzzy valuation function.

Consider $\mathscr{MF}$ a fuzzy model, $\mathcal{F} = \langle [0,1], T, S, N, I, B, 0, 1\rangle$ a fuzzy semantics and $w \in W$ a state. The notation, $[\![\mathscr{MF}, w \models_{\mathcal{F}} \varphi]\!]$ represents the **grade of uncertainty of a given formula** φ**, at state** w**, taking into account** $\mathscr{MF}$ **and** $\mathcal{F}$. The grade of uncertainty of $[\![\mathscr{MF}, w \models_{\mathcal{F}} \varphi]\!]$ is defined similarly in the way for a model.

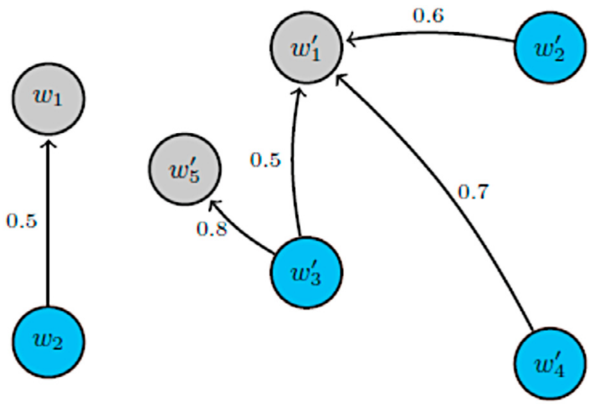
Notation 3: Given a relation $E \subset W \times W'$ and $w \in W$, we define:

a) $E[w] = \{w' \in W'; (w,w') \in E\}$;
b) $E^{-1}[w'] = \{w \in W; (w,w') \in E\}$.

Definition 5.7 *(Simulation [17]).* Let $\mathscr{MF} = \big(\langle W, R\rangle, V_{\langle W,R\rangle}\big)$ and $\mathscr{MF}' = \big(\langle W', R'\rangle, V_{\langle W',R'\rangle}\big)$ be two fuzzy models. A relation $E \subset W \times W'$ is said to be a **simulation** from $\mathscr{MF}$ to $\mathscr{MF}'$ if, for every $(w,w') \in E$:

1. $V_{\langle W,R\rangle}(w,p) \le V_{\langle W',R'\rangle}(w',p)$, for all $p \in \text{AtomProp}$.
2. For all $u \in W$; $R(w,u) \le \sup_{u' \in E[u]} R'(w',u')$.

Example 5.2. Consider the fuzzy models $(\mathscr{MF})_1 = \big(\langle W_1, R_1\rangle, V_1 = V_{\langle W_1,R_1\rangle}\big)$ and $(\mathscr{MF})_2 = \big(\langle W_2, R_2\rangle, V_2 = V_{\langle W_2,R_2\rangle}\big)$ in Fig. 19 s.t. $W_1 = \{w_1, w_2\}$, $W_2 = \{w'_1, w'_2, w'_3, w'_4, w'_5\}$ and $E = \{(w_1,w'_1), (w_1,w'_5), (w_2,w'_2), (w_2,w'_3), (w_2,w'_4)\}$. For all $p \in AtomProp$, consider $V_1(w_1,p) \le V_2(w'_1,p), V_1(w_1,p) \le V_2(w'_5,p), V_1(w_2,p) \le V_2(w'_2,p)$, $V_1(w_2,p) \le V_2(w'_3,p), V_1(w_2,p) \le V_2(w'_4,p)$.



(a) $(\mathscr{MF})_1$ (b) $(\mathscr{MF})_2$

Fig. 19. $(\mathscr{MF})_2$ simulates $(\mathscr{MF})_1$.

The relation $E \subset W_1 \times W_2$ is represented by the color of the nodes. If $(w, w') \in E$, then w and w' have the same color in the graph.

E is a simulation from $(\mathscr{MF})_1$ to $(\mathscr{MF})_2$. In fact, the condition 1 hold by assumption. To check the condition 2, we have to check for each pair in E.

Consider $w = w_1 \in W_1$. Therefore $u = w_2$ and $E[w_2] = \{w'_2, w'_3, w'_4\}$. We calculate,

- $R_1([w_1 w_1]) = 0 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_1 u'] = \sup\left\{R_2[w'_1 w'_1], R_2[w'_1 w'_5]\right\} = 0,$
- $R_1([w_1 w_1]) = 0 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_5 u'] = \sup\left\{R_2[w'_5 w'_1], R_2[w'_5 w'_5]\right\} = 0,$
- $R_1([w_1 w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_1 u'] = \sup\left\{R_2[w'_1 w'_2], R_2[w'_1 w'_3], R_2[w'_1 w'_4]\right\} = 0,$
- $R_1([w_1 w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_5 u'] = \sup\left\{R_2[w'_5 w'_2], R_2[w'_5 w'_3], R_2[w'_5 w'_4]\right\} = 0.$

Consider $w = w_2 \in W_1$. Therefore $u = w_1$ and $E[w_2] = \{w'_1, w'_5\}$. We calculate,

- $R_1([w_2 w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_2 u'] = \sup\left\{R_2[w'_2 w'_2], \{R_2[w'_2 w'_3], R_2[w'_2 w'_4]\right\} = 0,$
- $R_1([w_2 w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_3 u'] = \sup\left\{R_2[w'_3 w'_2], \{R_2[w'_3 w'_3], R_2[w'_3 w'_4]\right\} = 0,$
- $R_1([w_2 w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_4 u'] = \sup\left\{R_2[w'_4 w'_2], \{R_2[w'_4 w'_3], R_2[w'_4 w'_4]\right\} = 0,$
- $R_1([w_2 w_1]) = 0.5 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_2 u'] = \sup\left\{R_2[w'_2 w'_1], R_2[w'_2 w'_5]\right\} = 0.6,$
- $R_1([w_2 w_1]) = 0.5 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_3 u'] = \sup\left\{R_2[w'_3 w'_1], R_2[w'_3 w'_5]\right\} = 0.8,$
- $R_1([w_2 w_1]) = 0.5 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_4 u'] = \sup\left\{R_2[w'_4 w'_1], R_2[w'_4 w'_5]\right\} = 0.7.$

$(\mathscr{MF})_2$ simulates $(\mathscr{MF})_1$.

Definition 5.8 *(Bisimulation [17]).* Let $\mathscr{MF} = \left(\langle W, R\rangle, V_{\langle W,R\rangle}\right)$ and $\mathscr{MF}' = \left(\langle W', R'\rangle, V_{\langle W',R'\rangle}\right)$ be two fuzzy models. A relation $E \subset W \times W'$ is said to be a **bisimulation** from $\mathscr{MF}$ and $\mathscr{MF}'$ if, for every $(w, w') \in E$:

1. $V_{\langle W,R\rangle}(w, p) = V_{\langle W',R'\rangle}(w', p)$, for all $p \in$ AtomProp.
2. For all $u \in W$; $R(w, u) \leq \sup\limits_{u' \in E[u]} R'(w', u')$.
3. For all $u' \in W'$; $R'(w', u') \leq \sup\limits_{u \in E^{-1}[u']} R(w, u)$.

Example 5.3. Consider the fuzzy models $(\mathscr{MF})_1 = \left(\langle W_1, R_1\rangle, V_1 = V_{\langle W_1,R_1\rangle}\right)$ and $(\mathscr{MF})_2 = \left(\langle W_2, R_2\rangle, V_2 = V_{\langle W_2,R_2\rangle}\right)$ in Fig. 20 s.t. $W_1 = \{w_0, w_1, w_2\}$, $W_2 = \{w'_1, w'_2, w'_3, w'_4, w'_5\}$ and $E = \{(w_0, w'_1), (w_0, w'_5), (w_1, w'_1), (w_1, w'_5), (w_2, w'_2), (w_2, w'_3), (w_2, w'_4)\}$.

For all $p \in AtomProp$, consider $V_1(w_0, p) = V_2(w'_1, p)$; $V_1(w_0, p) = V_2(w'_5, p)$; $V_1(w_1, p) = V_2(w'_1, p)$; $V_1(w_1, p) = V_2(w'_5, p)$; $V_1(w_2, p) = V_2(w'_2, p)$; $V_1(w_2, p) = V_2(w'_3, p)$; $V_1(w_2, p) = V_2(w'_4, p)$.

As was done in the Example 5.2, the relation $E \subset W_1 \times W_2$ is represented by the color of the nodes. If $(w, w') \in E$, then w and w' have the same color in the graph.

In fact, the condition 1 hold by assumption.

To check the condition 2, once $E[w_0] = E[w_1] = \{w'_1, w'_5\}$ and $E[w_2] = \{w'_2, w'_3, w'_4\}$:

- $R_1([w_0w_0]) = 0 \leq \sup\limits_{u' \in E[w_0]} R_2[w'_1u'] = \sup\left\{R_2[w'_1w'_1], R_2[w'_1w'_5]\right\} = 0,$
- $R_1([w_0w_0]) = 0 \leq \sup\limits_{u' \in E[w_0]} R_2[w'_5u'] = \sup\left\{R_2[w'_5w'_1], R_2[w'_5w'_5]\right\} = 0,$
- $R_1([w_0w_1]) = 0 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_1u'] = \sup\left\{R_2[w'_1w'_1], R_2[w'_1w'_5]\right\} = 0,$
- $R_1([w_0w_1]) = 0 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_5u'] = \sup\left\{R_2[w'_5w'_1], R_2[w'_5w'_5]\right\} = 0,$
- $R_1([w_0w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_1u'] = \sup\left\{R_2[w'_1w'_2], R_2[w'_1w'_3], R_2[w'_1w'_4]\right\} = 0,$
- $R_1([w_0w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_5u'] = \sup\left\{R_2[w'_5w'_2], R_2[w'_5w'_3], R_2[w'_5w'_4]\right\} = 0,$
- $R_1([w_1w_0]) = 0 \leq \sup\limits_{u' \in E[w_0]} R_2[w'_1u'] = \sup\left\{R_2[w'_1w'_1], R_2[w'_1w'_5]\right\} = 0,$
- $R_1([w_1w_0]) = 0 \leq \sup\limits_{u' \in E[w_0]} R_2[w'_5u'] = \sup\left\{R_2[w'_5w'_1], R_2[w'_5w'_5]\right\} = 0,$
- $R_1([w_1w_1]) = 0 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_1u'] = \sup\left\{R_2[w'_1w'_1], R_2[w'_1w'_5]\right\} = 0,$
- $R_1([w_1w_1]) = 0 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_5u'] = \sup\left\{R_2[w'_5w'_1], R_2[w'_5w'_5]\right\} = 0,$
- $R_1([w_1w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_1u'] = \sup\left\{R_2[w'_1w'_2], R_2[w'_1w'_3], R_2[w'_1w'_4]\right\} = 0,$
- $R_1([w_1w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_5u'] = \sup\left\{R_2[w'_5w'_2], R_2[w'_5w'_3], R_2[w'_5w'_4]\right\} = 0,$
- $R_1([w_2w_0]) = 0.8 \leq \sup\limits_{u' \in E[w_0]} R_2[w'_2u'] = \sup\left\{R_2[w'_2w'_1], R_2[w'_2w'_5]\right\} = 0.8,$
- $R_1([w_2w_0]) = 0.8 \leq \sup\limits_{u' \in E[w_0]} R_2[w'_3u'] = \sup\left\{R_2[w'_3w'_1], R_2[w'_3w'_5]\right\} = 0.8,$
- $R_1([w_2w_0]) = 0.8 \leq \sup\limits_{u' \in E[w_0]} R_2[w'_4u'] = \sup\left\{R_2[w'_4w'_1], R_2[w'_4w'_5]\right\} = 0.8,$
- $R_1([w_2w_1]) = 0.5 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_4u'] = \sup\left\{R_2[w'_4w'_1], R_2[w'_4w'_5]\right\} = 0.8,$
- $R_1([w_2w_1]) = 0.5 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_2u'] = \sup\left\{R_2[w'_2w'_1], R_2[w'_2w'_5]\right\} = 0.8,$
- $R_1([w_2w_1]) = 0.5 \leq \sup\limits_{u' \in E[w_1]} R_2[w'_3u'] = \sup\left\{R_2[w'_3w'_1], R_2[w'_3w'_5]\right\} = 0.8,$
- $R_1([w_2w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_2u'] = \sup\left\{R_2[w'_2w'_2], R_2[w'_2w'_3], R_2[w'_2w'_4]\right\} = 0,$
- $R_1([w_2w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_3u'] = \sup\left\{R_2[w'_3w'_2], R_2[w'_3w'_3], R_2[w'_3w'_4]\right\} = 0,$
- $R_1([w_2w_2]) = 0 \leq \sup\limits_{u' \in E[w_2]} R_2[w'_4u'] = \sup\left\{R_2[w'_4w'_2], R_2[w'_4w'_3], R_2[w'_4w'_4]\right\} = 0.$

In order to check the condition 3, once $E^{-1}[w'_1] = E^{-1}[w'_5] = \{w_1, w_0\}$ and $E^{-1}[w'_2] = E^{-1}[w'_3] = E^{-1}[w'_4] = \{w_2\}$:

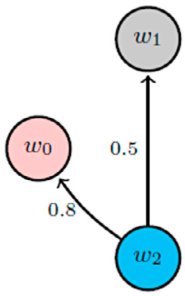
- $R_2([w'_1w'_1]) = 0 \leq \sup\limits_{u' \in E^{-1}[w'_1]} R_1[w_1u'] = \sup\{R_1[w_1w_1], R_1[w_1w_0]\} = 0,$
- $R_2([w'_1w'_1]) = 0 \leq \sup\limits_{u' \in E^{-1}[w'_1]} R_1[w_0u'] = \sup\{R_1[w_0w_1], R_1[w_0w_0]\} = 0,$
- $R_2([w'_1w'_2]) = 0 \leq \sup\limits_{u' \in E^{-1}[w'_2]} R_1[w_0u'] = \sup\{R_1[w_0w_2]\} = 0,$
- $R_2([w'_1w'_2]) = 0 \leq \sup\limits_{u' \in E^{-1}[w'_2]} R_1[w_1u'] = \sup\{R_1[w_1w_2]\} = 0,$
- $R_2([w'_1w'_3]) = 0 \leq \sup\limits_{u' \in E^{-1}[w'_3]} R_1[w_0u'] = \sup\{R_1[w_0w_2]\} = 0,$
- $R_2([w'_1w'_3]) = 0 \leq \sup\limits_{u' \in E^{-1}[w'_3]} R_1[w_1u'] = \sup\{R_1[w_1w_2]\} = 0,$
- $R_2([w'_1w'_4]) = 0 \leq \sup\limits_{u' \in E^{-1}[w'_4]} R_1[w_0u'] = \sup\{R_1[w_0w_2]\} = 0,$

- $R_2([w'_1 w'_4]) = 0 \leq \sup_{u' \in E^{-1}[w'_4]} R_1[w_1 u'] = \sup\{R_1[w_1 w_2]\} = 0,$
- $R_2([w'_1 w'_5]) = 0 \leq \sup_{u' \in E^{-1}[w'_5]} R_1[w_1 u'] = \sup\{R_1[w_1 w_1], R_1[w_1 w_0]\} = 0,$
- $R_2([w'_1 w'_5]) = 0 \leq \sup_{u' \in E^{-1}[w'_5]} R_1[w_0 u'] = \sup\{R_1[w_0 w_1], R_1[w_0 w_0]\} = 0,$
- $R_2([w'_2 w'_1]) = 0.8 \leq \sup_{u' \in E^{-1}[w'_1]} R_1[w_2 u'] = \sup\{R_1[w_2 w_1], R_1[w_2 w_0]\} = 0.8,$
- $R_2([w'_2 w'_2]) = 0 \leq \sup_{u' \in E^{-1}[w'_2]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_2 w'_3]) = 0 \leq \sup_{u' \in E^{-1}[w'_3]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_2 w'_4]) = 0 \leq \sup_{u' \in E^{-1}[w'_4]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_2 w'_5]) = 0 \leq \sup_{u' \in E^{-1}[w'_5]} R_1[w_2 u'] = \sup\{R_1[w_2 w_1], R_1[w_2 w_0]\} = 0.8,$
- $R_2([w'_3 w'_1]) = 0.8 \leq \sup_{u' \in E^{-1}[w'_1]} R_1[w_2 u'] = \sup\{R_1[w_2 w_1], R_1[w_2 w_0]\} = 0.8,$
- $R_2([w'_3 w'_2]) = 0 \leq \sup_{u' \in E^{-1}[w'_2]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_3 w'_3]) = 0 \leq \sup_{u' \in E^{-1}[w'_3]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_3 w'_4]) = 0 \leq \sup_{u' \in E^{-1}[w'_4]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_3 w'_5]) = 0.7 \leq \sup_{u' \in E^{-1}[w'_5]} R_1[w_2 u'] = \sup\{R_1[w_2 w_1], R_1[w_2 w_0]\} = 0.8,$
- $R_2([w'_4 w'_1]) = 0.8 \leq \sup_{u' \in E^{-1}[w'_1]} R_1[w_2 u'] = \sup\{R_1[w_2 w_1], R_1[w_2 w_0]\} = 0.8,$
- $R_2([w'_4 w'_2]) = 0 \leq \sup_{u' \in E^{-1}[w'_2]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_4 w'_3]) = 0 \leq \sup_{u' \in E^{-1}[w'_3]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_4 w'_4]) = 0 \leq \sup_{u' \in E^{-1}[w'_4]} R_1[w_2 u'] = \sup\{R_1[w_2 w_2]\} = 0,$
- $R_2([w'_4 w'_5]) = 0 \leq \sup_{u' \in E^{-1}[w'_5]} R_1[w_2 u'] = \sup\{R_1[w_2 w_1], R_1[w_2 w_0]\} = 0.8,$
- $R_2([w'_5 w'_1]) = 0 \leq \sup_{u' \in E^{-1}[w'_1]} R_1[w_1 u'] = \sup\{R_1[w_1 w_1], R_1[w_1 w_0]\} = 0,$
- $R_2([w'_5 w'_1]) = 0 \leq \sup_{u' \in E^{-1}[w'_1]} R_1[w_0 u'] = \sup\{R_1[w_0 w_1], R_1[w_0 w_0]\} = 0,$
- $R_2([w'_5 w'_2]) = 0 \leq \sup_{u' \in E^{-1}[w'_2]} R_1[w_0 u'] = \sup\{R_1[w_0 w_2]\} = 0,$
- $R_2([w'_5 w'_2]) = 0 \leq \sup_{u' \in E^{-1}[w'_2]} R_1[w_1 u'] = \sup\{R_1[w_1 w_2]\} = 0,$
- $R_2([w'_5 w'_3]) = 0 \leq \sup_{u' \in E^{-1}[w'_3]} R_1[w_0 u'] = \sup\{R_1[w_0 w_2]\} = 0,$
- $R_2([w'_5 w'_3]) = 0 \leq \sup_{u' \in E^{-1}[w'_3]} R_1[w_1 u'] = \sup\{R_1[w_1 w_2]\} = 0,$
- $R_2([w'_5 w'_4]) = 0 \leq \sup_{u' \in E^{-1}[w'_4]} R_1[w_0 u'] = \sup\{R_1[w_0 w_2]\} = 0,$
- $R_2([w'_5 w'_4]) = 0 \leq \sup_{u' \in E^{-1}[w'_4]} R_1[w_1 u'] = \sup\{R_1[w_1 w_2]\} = 0,$
- $R_2([w'_5 w'_5]) = 0 \leq \sup_{u' \in E^{-1}[w'_5]} R_1[w_1 u'] = \sup\{R_1[w_1 w_1], R_1[w_1 w_0]\} = 0,$
- $R_2([w'_5 w'_5]) = 0 \leq \sup_{u' \in E^{-1}[w'_5]} R_1[w_0 u'] = \sup\{R_1[w_0 w_1], R_1[w_0 w_0]\} = 0.$

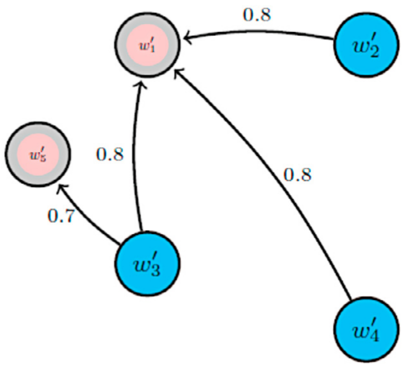
There is a bisimulation between $(\mathscr{MF})_1$ and $(\mathscr{MF})_2$.

In order to define the bisimulation for $RFSG$s, we will present a sequence of results presented in [22].

Lema 5.1. *[17] Given fuzzy models $\mathscr{MF} = (\langle W, R\rangle, V_{\langle W,R\rangle})$ and $\mathscr{MF}' = (\langle W', R'\rangle, V_{\langle W',R'\rangle})$ with the Gödel semantics and a bisimulation $E \subset W \times W'$ s.t. $(w, w') \in E$. Then*



(a) $(\mathscr{MF})_1$



(b) $(\mathscr{MF})_2$

Fig. 20. There is a bisimulate between $(\mathscr{MF})_1$ and $(\mathscr{MF})_2$.

$$[\![\mathscr{MF}, w \models_{\mathcal{G}} \varphi]\!] = [\![\mathscr{MF}', w' \models_{\mathcal{G}} \varphi]\!]$$

for every formula.

Definition 5.9. Let us consider the models $\mathscr{M} = (M, V_M)$ and $\mathscr{M}' = (M', V_{M'})$ ($M = \langle W, \mu\rangle$ and $M' = \langle W', \mu'\rangle$ are $RFSGs$) and the relation $E \subset W \times W'$. Given the induced fuzzy models $\tilde{M} = \big(\langle \tilde{W}, \tilde{R}\rangle, V_{\langle \tilde{W}, \tilde{R}\rangle}\big)$ and $\tilde{M}' = \big(\langle \tilde{W}', \tilde{R}'\rangle, V_{\langle \tilde{W}', \tilde{R}'\rangle}\big)$, the relation $\tilde{E} \subset \tilde{W} \times \tilde{W}'$ is an **extension** of E if $\big((w, \mu), (w', \mu')\big) \in \tilde{E}$ whenever $(w, w') \in E$.

Definition 5.10. Given two $RFSGs$ $M = \langle W, \mu\rangle$ and $M' = \langle W', \mu'\rangle$, a relation $E \subseteq W \times W'$ is a **bisimulation** between the models $\mathscr{M} = (M, V_M)$ and $\mathscr{M}' = (M', V_{M'})$, if there is an extension $\tilde{E}$ which is a bisimulation between the induced fuzzy models $\tilde{\mathscr{MF}}$ and $\tilde{\mathscr{MF}}'$.

Theorem 5.2. *Given the $RFSGs$ $M = \langle W, \mu\rangle$ and $M' = \langle W, \mu\rangle$ with the aggregation A and a bisimulation $E \subset W \times W'$. If $(w, w') \in E$, considering the models $\mathscr{M} = (M, V_M)$, $\mathscr{M}' = (M', V_{M'})$ and the Gödel Semantic $\mathcal{G}$, then*

$$[\![\mathscr{M}, w \models^A_{\mathcal{G}} \varphi]\!] = [\![\mathscr{M}', w' \models^A_{\mathcal{G}} \varphi]\!]$$

for every formula $\varphi \in AtomProp$.

Proof. By Definition 5.10, there is a bisimulation $\tilde{E}$ between the induced fuzzy models $\tilde{\mathscr{MF}} = \big(\langle W, R\rangle, V_{\langle W, R\rangle}\big)$ and $\tilde{\mathscr{MF}}' = \big(\langle W', R'\rangle, V_{\langle W', R'\rangle}\big)$. By the Lema 5.1 and the Theorem 5.1,

$$[\![\mathscr{M}, w \models^A_{\mathcal{G}} \varphi]\!] = [\![\tilde{\mathscr{MF}}, (w, \mu) \models_{\mathcal{G}} \varphi]\!] = [\![\tilde{\mathscr{MF}}', (w', \mu') \models_{\mathcal{G}} \varphi]\!] = [\![\mathscr{M}', w' \models^A_{\mathcal{G}} \varphi]\!]$$

6. Modeling a tank level control system

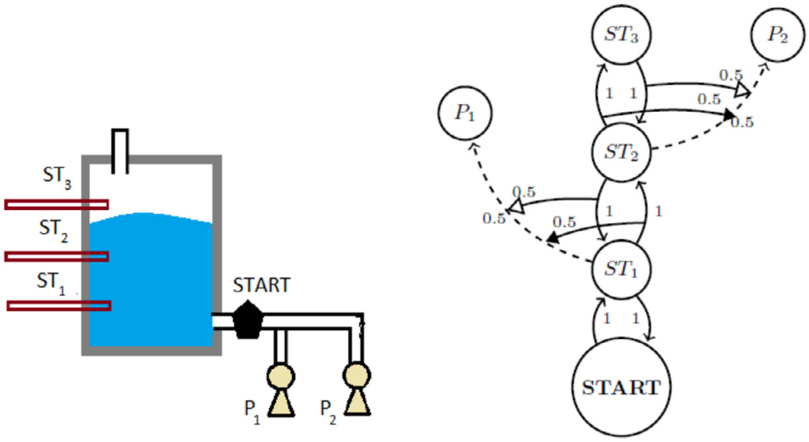
In industrial processes that use tanks, the control of the fluid level is a common practice. Even with a relatively simple structure, logic controllers are often used. The study and the modeling of tank plants and logic controllers are important because they provide the understanding of the current scenario of the system, causing benefits such as: the increase of productivity and the prevention of accidents [11].

Fig. 21 (a) shows a scheme where a tank control system is built with three signal transmitters $\{ST_1, ST_2, ST_3\}$, two pumps $\{P_1, P_2\}$ and a channel for fluid inlet called $START$. The dynamics of the system works as follows:

- **Fluid level rising**: At $START$ the fluid starts to be inserted into the tank while pumps P_1 and P_2 are on standby receiving a minimum electric current. When the fluid level triggers ST_2, P_1 receives an increment of electric current and is activated. If the fluid level continues to rise and trigger ST_3, the pump P_2 receives an increment of electric current and is also activated.
- **Fluid level decreasing**: When the fluid level is maximum, the pumps P_1 and P_2 are active. When the fluid level decreases, the ST_3 is triggered and P_2 goes to standby with a decrease in its electric current. If the fluid level continues to decrease, ST_2 is triggered and P_1 goes to standby with a decreasing in its electric current.

The signal transmitter receives the difference pressure of two points with different weights and converts it into a proportional electrical signal. This electric signal is sent to pumps [11].

Consider in Fig. 21(b):



(a) Tank Scheme (b) $RFRG\ M_R$

Fig. 21. Model of tank control system.

- The set of arrows S;
- The set of worlds $W = \{ST_1, ST_2, ST_3, P_1, P_2, START\}$;
- The membership function $\mu : S \to [0,1] \times \{\text{ON}, \text{OFF}\}$ which assign to each arrow in S, the electric signal generated when they are crossing;
- The function $A_g : S_\to \to A$, where $A = \{T_L, S_L\}$.

The $RFRG$ $M_R = \langle M, A_g \rangle$ models the system of tank control above. These systems could also be model by using a FSG in which all arrows are active and all high-order arrows are connecting. However, in this case, there would be no possibility of working on deactivation of the pumps.

The reconfiguration of $\mathcal{M}_R$, after crossing the arrows sequence $[ST_1\ ST_2]$, $[ST_1\ ST_2][ST_2\ ST_3]$ and $[ST_1\ ST_2][ST_2\ ST_3][ST_3\ ST_2]$, can be observed in Fig. 22. Assuming:

- $\mu(a_i^0) = 1$, for all $a_i^0 \in S^0 - \{[ST_1\ P_1], [ST_2\ P_2]\}$;
- $A_g([ST_1\ ST_2]) = A_g([ST_2\ ST_3]) = S_L$;
- $A_g([ST_2\ ST_1]) = A_g([ST_3\ ST_2]) = T_L$.

The fuzzy value and the status of the arrow $[ST_1\ P_1]$ after the arrow $[ST_1\ ST_2]$ has been crossed is calculate in the follow way:

$$\begin{aligned}
\mu^{Ag}_{[ST_1\ ST_2]}([ST_1\ P_1]) &= \Big(S_L\big(1, 0.5, 0.5\big), \text{ON} \Big) \\
&= \Big(S_L\Big(1, S_L\big(0.5, 0.5\big)\Big), \text{ON} \Big) \\
&= \Big(S_L(1, 1), \text{ON} \Big) \\
&= \Big(1, \text{ON} \Big)
\end{aligned}$$

Consider the propositions

$$p\text{:“}P_1 \textit{ is active}\text{” and } q\text{:“}P_2 \textit{ is active}\text{”}$$

for the model $\mathscr{M}_R = \langle M_R, V \rangle$, with $V(ST_1, p) = 0.05$, $V(ST_2, p) = 0.08$, $V(ST_3, p) = 0.6$, $V(ST_1, q) = 0.01$, $V(ST_2, q) = 0.5$ and $V(ST_3, q) = 0.7$. Using the *Gödel semantics*, we are able to compute the grade of uncertainty of the formula $S_{Next}(p \wedge q)$ to the states ST_2 and ST_3:

$$[\![\mathscr{M}_R, ST_2 \models^{A_g}_{\mathcal{F}_G} S_{Next}(p \wedge q)]\!] = \mathbf{S}_M\Big(\mathbf{T}_M\Big(1, [\![\mathscr{M}_R^{[ST_2\ ST_1]}, ST_1 \models^{A_g}_{\mathcal{F}_G} (p \wedge q)]\!]\Big), \mathbf{T}_M\Big(1, [\![\mathscr{M}_R^{[ST_2\ ST_3]}, ST_3 \models^{A_g}_{\mathcal{F}_G} (p \wedge q)]\!]\Big)\Big) = 0.6$$

and $[\![\mathscr{M}_R, ST_3 \models^{A_g}_{\mathcal{F}_G} S_{Next}(p \wedge q)]\!] = \mathbf{S}_M\Big(\mathbf{T}_M\Big(1, [\![\mathscr{M}_R^{[ST_3\ ST_2]}, ST_2 \models^{A_g}_{\mathcal{F}_G} (p \wedge q)]\!]\Big)\Big) = 0.08.$

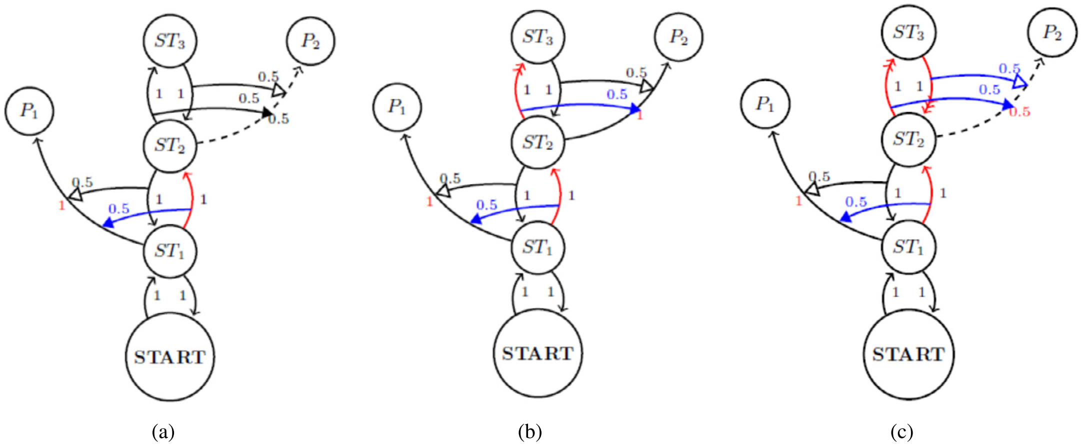


Fig. 22. M_R configuration after: (a) $[ST_1\ ST_2]$, (b) $[ST_1\ ST_2][ST_2\ ST_3]$ and (c) $[ST_1\ ST_2][ST_2\ ST_3][ST_3\ ST_2]$.

So, the degree of the sentence,

"*There is a next state in which the pumps P1 and P2 are working*"

at the state ST_2 is 0.6 and at the state ST_3 is 0.08.

7. Final remarks

Reversal Fuzzy Switch Graphs ($RFSG$) are structures designed to model reactive systems which provide the activation and deactivation of resources. This paper presents, with more details, the $RFSG$s as well as the operations presented in [7].

The valuation of the membership function can occur on any lattice, however, depending on this choice, the resulting formal logic must be adjusted. For example, if we consider the lattice of intervals with the Kulish-Miranker order, considering correctness, then the modal logic associated with the graph will have a non-residual implication [23].

As a first new contribution in relation to [7], we present the concept of fuzzy induced graph from a $RFSG$. It is a connection between $RFSG$s and fuzzy graphs which allows a finite representation for infinite fuzzy graphs. The attribution of aggregations in this relationship, however, has not been explored and will be the subject of further studies. Still on this topic, it was presented a recursive method for constructing, from a finite base set, an induced graph. In future works, we intend to relate the base set of a induced fuzzy graph to this original $RFSG$.

Another new concept presented in this paper was the simulation and bisimulation of $RFSG$s. These notions, however, were established from the concept of model. Other types of logics and other notions of certainty that allow to define the bisimulation between $RFSG$s more directly will be subject of future works.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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