Reversal Fuzzy Switch Graphs

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Abstract. Fuzzy Switch Graph (FSG) generalize the notion of Fuzzy Graph by adding high-order arrows and aggregation functions which update the fuzzy values of arrows whenever a zero-order arrow is crossed. In this paper, we propose a more general structure called Reversal Fuzzy Switch Graph ($RFSG$), which promotes other actions in addition to updating the fuzzy values of the arrows, as activation and deactivation of the arrows. $RFSG$s are able to model dynamical aspects of some systems which generally appear in: Engineering, Computer Science and some other fields. The paper also provides a logic to verify properties of the modelled system and closes with an application.

Keywords: Reversal Fuzzy Switch Graphs · Reversal Fuzzy Reactive Graphs · Fuzzy Switch Graphs · Fuzzy Systems · Reactive Systems

1 Introduction

Reactive graphs are structures whose the relations change when we move along the graph. This concept has been introduced by Dov Gabbay in 2004 (see [8],[7]) and generalizes the static notion of a graph by incorporating of high-order edges (called high-order arrows or switches). Graphs with these characteristics are called *Switch Graphs*.

In [13], Santiago et al. introduce the notion of *Fuzzy Switchs Graphs* (FSGs). These graphs are able to model reactive systems endowed with fuzziness and extend the notion of fuzzy graphs, in the sense that crossing an edge (zero-order arrow) induces an update of the system using high-order arrows and aggregation functions. For systems which require different aggregations for updating different arrows, Santiago et al. [13] introduced the *Fuzzy Reactive Graphs* (FRGs).

FSGs and FRGs, however, are not sufficient to model systems in which other edges of the system are activated or deactivated when one edge is crossing. To incorporate this, in this paper, we propose the notion of *Reversal Fuzzy Switch Graphs* ($RFSG$s). We also introduce the cartesian product of $RFSG$s and a logic to verify properties of such structures. We close this paper by showing an application of $RFSG$s.

The paper is organized as follows: Section 2 presents some basic concepts. Section 3 introduces the notion of $RFSG$s, how they can be used to model the reactivity of some fuzzy systems and presents some algebraic operations. Section 4 provides a logic for $RFSG$s. Section 5 shows explains how $RFSG$s can be used to model a dynamic control system. Finally, section 6 provides some final remarks.

2 Preliminaries

In this section we recall some concepts and results found in the literature in order to make this paper self-contained. We assume that the reader has some basic knowledge in fuzzy set theory.

Definition 1 ([10]). *A **fuzzy set** A, defined on a non-empty set X, is characterized by a **membership function** $\varphi_A : X \to [0,1]$. The value $\varphi_A(x) \in [0,1]$ measures the degree of membership of x in the set A.*

Definition 2 (Fuzzy Graphs [10]). *A **fuzzy graph** is a structure $\langle V, R\rangle$, such that V is a non-empty set called **set of vertices** and R is a fuzzy set $R : V \times V \to [0,1]$.*

For simplicity, we assume the set of vertices is a crisp set, in contrast to what is defined as a fuzzy graph in [10]. The Figure 1(a) shows a fuzzy graph.

Dov Gabbay [2] provided graphs with high-order arrows in order to model reactive behaviors. This kind of graphs is defined as follows.

Definition 3 (Switch Graphs [2] [6]). *A **switch graph** is a pair $\langle W, R\rangle$ s.t. W is a non-empty set (set of **worlds**) and $R \subseteq A(W)$ is a set of arrows, called **switches** or **high-order arrows**, where $A(W) = \bigcup_{i\in\mathbb{N}} A_i(W)$ with*

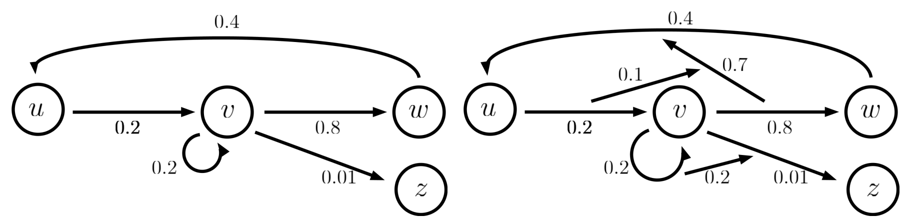
$$\begin{cases} A_0(W) = W \times W \\ A_{i+1}(W) = A_0(W) \times A_i(W) \end{cases} \tag{1}$$

Fuzzy Switch Graphs were introduced by Santiago et al. in [13].

Definition 4 (Fuzzy Switch Graphs [13]). *Let W be a non-empty finite set (set of **states** or **worlds**) and the family of sets $S = \bigcup_{n\in\mathbb{N}} S^n$ where,*

$$\begin{cases} S^0 \subseteq W \times W \\ S^{n+1} \subseteq S^0 \times S^n \end{cases} \tag{2}$$

*A **fuzzy switch graph (FSG)** is a pair $\mathcal{M} = \langle W, \varphi : S \to [0,1]\rangle$, where φ is a fuzzy subset of S. The elements $a_i^0 \in S^0 (i \in \mathbb{N})$ are called **zero-order arrows**. The elements of S^{n+1} are called **high-order arrows**.*



(a) Fuzzy Graph (b) Fuzzy Switch Graph

Fig. 1. Santiago et al.(2020) [13].

Example 1. The Figure 1 shows a fuzzy graph and a fuzzy switch graph.

Fuzzy Logic provides many proposals for logical connectives. In what follows we rewiew the notions of fuzzy conjunctions, disjunctions, implications and negations. The first two are generalized by t-norms and t-conorms, respectively [9].

Definition 5 (t-norms and t-conorms). *A* ***uninorm*** *function is a bivariate function* $U : [0,1] \times [0,1] \to [0,1]$*, that is isotonic, commutative, associative with a neutral element* $e \in [0,1]$*. If* $e = 1$*, then* U *is called* ***t-norm*** *and if* $e = 0$*, then* U *is called* ***t-conorm****.*

Example 2. The functions $T_G(x,y) = min(x,y)$ and $T_L(x,y) = max(x+y-1,0)$ (*Łukasiewicz*) are t-norms. The functions $S_G(x,y) = max(x,y)$ and $S_L(x,y) = min(x+y,1)$ (*Łukasiewicz*) are t-conorms.

Notation*:* Let T be a t-norm, $f : [0,1] \to [0,1]$ and J_n a finite subset of $[0,1]$ with n elements ($J_0 = \emptyset$). We define $\underset{a \in J_n}{T}$ in this way,

$$\underset{a \in J_n}{T} f(a) = \begin{cases} 1, \text{ case } n = 0; \\ f(a), \text{ case } n = 1; \\ T\big(x, \underset{a \in J_m}{T} f(a)\big), \text{ case } n > 1, x \in J_n \text{ and } J_m = J_n \backslash \{x\}. \end{cases} \tag{3}$$

Similarly, for S t-conorm, we define $\underset{a \in J_n}{S}$ s.t.

$$\underset{a \in J_n}{S} f(a) = \begin{cases} 0, \text{ case } n = 0; \\ f(a), \text{ case } n = 1; \\ S\big(x, \underset{a \in J_m}{S} f(a)\big), \text{ case } n > 1, x \in J_n \text{ and } J_m = J_n \backslash \{x\}. \end{cases} \tag{4}$$

Note that, since T and S are comutative and associative, $\underset{a \in J_m}{T}$ and $\underset{a \in J_m}{S}$ are well defined. That is, it does not depend on the way we choice $x \in J_n$ to make $J_n = \{x\} \cup J_m$.

Example 3. Given the t-norm $T(x,y) = min(x,y)$, the identity function $Id : [0,1] \to [0,1]$ and the set $J_3 = \{x_1, x_2, x_3\} \subset [0,1]$, we have:

$$\underset{a\in J_3}{T} Id(a) = min\Big(x_1, \underset{a\in J_2}{T} Id(a)\Big) = min\Big(x_1, min\big(x_2, \underset{a\in J_1}{T} Id(a)\big)\Big) =$$
$$min\Big(x_1, min\big(x_2, Id(x_3)\big)\Big) = min\Big(x_1, min\big(x_2, x_3\big)\Big).$$

Definition 6 (Negations [1]). *A unary operation $N : [0,1] \to [0,1]$ is a **fuzzy negation** if $N(0) = 1, N(1) = 0$ and N is decreasing.*

Example 4. Gödel Negation: $N_G : [0,1] \to [0,1]$ s.t. $N_G(0) = 1$ and $N_G(x) = 0$, whenever $x > 0$.

Definition 7 (Implications [1]). *A bivariate function $I : [0,1]^2 \to [0,1]$ is a **fuzzy implication** if it is decreasing with respect to the first variable, increasing with respect to the second variable, $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$ (boundary conditions).*

Example 5. Gödel Implication: $I_G : [0,1]^2 \to [0,1]$ s.t. $I_G(x,y) = 1$, whenever $x \leq y$, and $I_G(x,y) = y$ otherwise.

Definition 8 (Bi-implications [11]). *A bivariate function $B : [0,1]^2 \to [0,1]$ is a **fuzzy bi-implication** if it is commutative, $B(x,x) = 1$, $B(0,1) = 0$ and $B(w,z) \leq B(x,y)$, whenever $w \leq x \leq y \leq z$.*

Example 6. Gödel Bi-implication: $B_G(x,y) = T_G(I_G(x,y), I_G(y,x))$.

Definition 9 (Fuzzy Semantics [4]). *A structure $\mathcal{F} = \{[0,1], T, S, N, I, B, 0, 1\}$, s.t. T is a t-norm, S is a t-conorm, N a fuzzy negation, I is a fuzzy implication, and B a fuzzy bi-implication is a called **fuzzy semantics**.*

Example 7. Gödel Semantic: $\mathcal{G} = \{[0,1], T_M, S_M, N_G, I_G, B_G, 0, 1\}$.

Aggregation functions are functions with special properties which generalize the means, like *arithmetic mean*, *weighted mean* and *geometric mean.*

Definition 10 (Aggregation Function [3]). *An **aggregation function** is a n-ary function $A : [0,1]^n \to [0,1]$, with $A(0,0,...,0) = 0$, $A(1,1,...,1) = 1$ and, for all $\bar{x}, \bar{y} \in [0,1]^n$, $\bar{x} \leq \bar{y}$ implies $A(\bar{x}) \leq A(\bar{y})$.*

Example 8. $A_n(\bar{x}) = \dfrac{1}{n}(x_1 + ... + x_n)$ (*Arithmetic mean*), $A_n(\bar{x}) = \sqrt[n]{x_1 + ... + x_n}$ (*Geometric mean*), t-norms, t-conorms and projection functions, $\pi_j : A_1 \times ... \times A_j \times ... \times A_n \longrightarrow A_j$, s.t. $\pi_j(x_1, ..., x_j, ..., x_n) = x_j$, are aggregation functions.

Definition 11 ([3]). *For every $\bar{x} \in [0,1]^n$, an aggregation function A is, **averaging** if $min(\bar{x}) \leq A(x) \leq max(\bar{x})$, **conjunctive** if $A(x) \leq min(\bar{x})$ and **disjunctive** if $A(x) \geq max(\bar{x})$.*

Example 9. t-norms are conjunctive aggregations, t-conorms are disjunctive and means (*arithmetic, geometric, weighted*) are average aggregations. For example, given $x, y \in [0,1]$ we have:

$$xy \leq min\{x,y\} \leq \frac{x+y}{2} \leq max\{x,y\}.$$

Definition 12. *An aggregation $A : [0,1]^n \to [0,1]$ is **shift-invariant** if, for all $\lambda \in [-1,1]$ and for all $\bar{x} \in [0,1]^n$,*

$$A(x_1 + \lambda, ..., x_n + \lambda) = A(x_1, ..., x_n) + \lambda$$

whenever $(x_1 + \lambda, ..., x_n + \lambda) \in [0,1]^n$ and $A(x_1, ..., x_n) + \lambda \in [0,1]$.

Definition 13. *An element $a \in \,]0,1[$ is a **zero divisor** of an aggregation A if, for all $i \in \{1, ..., n\}$, there is some $\bar{x} \in \,]0,1]^n$ such that its i-th component is $x_i = a$ and $A(\bar{x}) = 0$, i.e., the equality $A(x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_n) = 0$ holds.*

Example 10. The function $A(x_1, x_2) = max(0, x_1 + x_2 - 1)$ is an aggregation with zero divisor $a = 0.999$.

In this paper, we avoid aggregations with zero divisors, since it will induce the disconnection of edges.

In [13], Santiago et al. extend the notion of FSGs for *Fuzzy Reactive Graphs*.

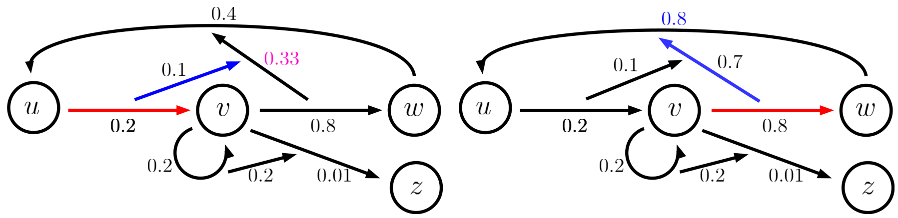
Definition 14 (Fuzzy Reactive Graphs). *Let $\mathcal{M} = \langle W, \varphi : S \to [0,1] \rangle$ be a FSG, the set of zero-order arrows $\Gamma_{\mathcal{M}} = \{a_i^0 \in S^0 : \varphi(a_i^0) > 0\}$, $A_{\mathcal{M}} = \{A_1, ..., A_k : [0,1]^3 \to [0,1]\}$ a set of aggregation functions and a function $Ag_{\mathcal{M}} : \Gamma_{\mathcal{M}} \to A_{\mathcal{M}}$. The pair $\mathcal{M}_R = \langle \mathcal{M}, Ag_{\mathcal{M}} \rangle$ is called a **fuzzy reactive graph (FRG)**.*

Example 11. Let $\mathcal{M}$ be the FSG in Figure 1(b). Consider $\Gamma_{\mathcal{M}} = \{a_1^0 = (u,v), a_2^0 = (v,v), a_3^0 = (v,w), a_4^0 = (v,z), a_5^0 = (w,u)\}$ and $A_{\mathcal{M}} = \{arith,\ max\}$. Defining the application $Ag_{\mathcal{M}} : \Gamma_{\mathcal{M}} \to A_{\mathcal{M}}$ s.t. $Ag(a_1^0) = Ag(a_2^0) = arith$ and $Ag(a_3^0) = Ag(a_4^0) = Ag(a_5^0) = max$, we have the FRG $\mathcal{M}_R = \langle \mathcal{M}, Ag \rangle$. The Figure 2(a) contains the reconfiguration of $\mathcal{M}_R$ after crossing $a_1^0 = (u,v)$ and having applied $Ag(a_1^0) = arith$ to the fuzzy values: 0.2, 0.1, 0.7. The Figure 2(b) contains the reconfiguration of M_R after crossing $a_3^0 = (v,w)$ and having aplied $Ag(a_3^0) = max$ to the fuzzy values: 0.8, 0.7, 0.4.

3 Reversal Fuzzy Switch Graphs

In this section we introduce the notion of Reversal Fuzzy Switch Graph, a structure which generalizes the notion of Fuzzy Switch Graph introduced by Santiago et al. [13]. This new kind of graph has in its structure two new types of high order-arrows, called connection arrows and disconnecting arrows. These arrows allow to model reactive systems in which the accessibility to the worlds may be activated or deactivated by the transitions.

In what follows W and V are non-empty finite sets.



(a) (b)

Fig. 2. Santiago et al.(2020) [13].

Definition 15 (Reversal Fuzzy Switch Graphs). *Let W be a set whose elements are called **states** or **words**. Consider the following family of sets defined recursively,*

$$\begin{cases} S^0 \subseteq W \times W \\ S^{n+1} \subseteq S^0 \times S^n \times \{\bullet, \circ\} \end{cases} \tag{5}$$

*and $S = \bigcup_{n \in \mathbb{N}} S^n$. A **reversal fuzzy switch graph (RFSG)** is a pair $M = \langle W, \mu : S \to [0,1] \times \{\text{ON}, \text{OFF}\}\rangle$. Arrows with $\bullet$ in their third component are called **connecting arrows** and arrows with $\circ$ in their third component are called **disconnecting arrows**. When the context is clear we denote a RFSG simply by $\langle W, \mu\rangle$.*

Active arrows are drawn with a normal line whereas inactive arrows are drawn with a dashed line. Moreover, connecting (disconnecting) arrows change the targeted arrow state for active (inactive) and are drawn with a black (white) arrowhead.

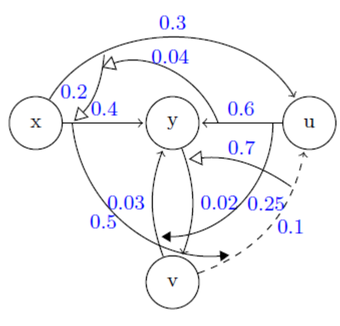


Fig. 3. *RFSG* with fuzzy values

For readability, we introduce some notation and nomenclatures:

- Arrows in S^n will be denoted by a_i^n, for $n \geq 0$ and $i \in \mathbb{N}$. Connecting (disconnecting) arrows receive an $\bullet(\circ)$ marker above. In this context, for $i \in \mathbb{N}$, a_i^0 are used to represent **zero-order arrows**, $\mathring{a}_i^1$ are are used to represent **disconnecting one-order arrows**, $\dot{a}_i^1$ are used to represent **connecting one-order arrows**, $\mathring{a}_i^2$ are used to represent **disconnecting two-order arrows**, $\dot{a}_i^2$ are used to represent **connecting two-order arrows**, and so on.
- In the following, we make an abuse of notation. When necessary and if the context is clear, we will denote in more detail the arrows in S^n in a more expanded way. For example, a_i^0 from x to y will be denoted by $[xy]$, the disconnecting and connecting one-order arrows $\mathring{a}_i^1$ and $\dot{a}_i^1$, from $[xy]$ to $[uv]$ will be denoted by $[\![[xy],[uv],\circ]\!]$ and $[\![[xy],[uv],\bullet]\!]$, respectively. When referring to any high-order arrow, we write $\sigma \in \{\circ,\bullet\}$ instead of $\circ$ or $\bullet$. For example, any one-order arrow from $[uv]$ to $[xy]$ will be written $[\![[uv],[xy],\sigma]\!]$. Any two-order arrows from $[zw]$ to $[\![[xy],[uv],\sigma]\!]$ will be enoted by $[\![[zw],[\![[xy],[uv],\sigma]\!],\sigma]\!]$.
- When there is no need to specify the order of the arrow belonging to set S, we will denote $a \in S$.
- Given the projection functions $\pi_1 : [0,1] \times \{\text{ON},\text{OFF}\} \to [0,1]$ and $\pi_2 : [0,1] \times \{\text{ON},\text{OFF}\} \to \{\text{ON},\text{OFF}\}$, if $a \in S$ we write $\mu_1(a) = \pi_1(\mu(a))$ and $\mu_2(a) = \pi_2(\mu(a))$ to indicate the first and second components of $\mu(a)$.

In the following, we will consider the $RFSG$s $M = \langle W,\mu\rangle$ and $M' = \langle W,\mu'\rangle$.

Definition 16. *M is a **subgraph** of M' if $\mu_1(a) \leq \mu_1'(a)$ and $\mu_2(a) = \mu_2'(a)$, for all $a \in S$. M is a **supergraph** of M' if $\mu_1(a) \geq \mu_1'(a)$ and $\mu_2(a) = \mu_2'(a)$, for all $a \in S$.*

Definition 17. *M' is a **translation** of M by $\lambda \in [-1,1]$ if, for all $a \in S$ s.t. $\mu_1(a) > 0$, $\mu_1'(a) = \mu_1(a) + \lambda \in [0,1]$ and $\mu_2'(a) = \mu_2(a)$.*

3.1 Reactivity of *RFSG*s

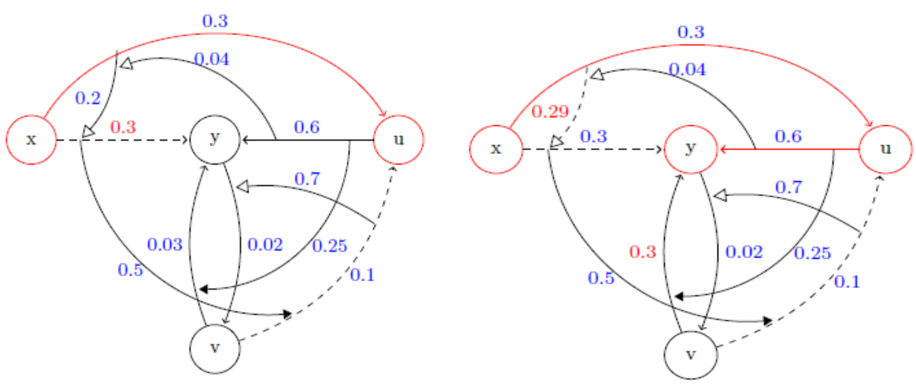
Intuitively, a reactive graph is a graph that may change its configuration when a zero-order arrow is crossed. In order to model this global dependence in a $RFSG$, we consider the reactivity idea presented in [13] with the necessary adaptations: *Whenever a zero-order arrow is crossed, the fuzzy value and the arrow state (active or inactive) of its target arrows are updated.*

Definition 18. *Given M and an aggregation function $A : [0,1]^3 \to [0,1]$, a $RFSG$ **based on** A **after crossing a zero-order arrow** a_i^0, is the RFSG $M_{a_i^0}^A = \langle W, \mu_{a_i^0}^A : S \to [0,1] \times \{\text{ON},\text{OFF}\}\rangle$ s.t.*

$$\mu_{a_i^0}^A(a) = \begin{cases} \Big(A\big(\mu_1(a_i^0),\mu_1([\![a_i^0,a,\bullet]\!]),\mu_1(a)\big),\text{ON}\Big), & \textit{if } [\![a_i^0,a,\bullet]\!] \in S; \\ \Big(A\big(\mu_1(a_i^0),\mu_1([\![a_i^0,a,\circ]\!]),\mu_1(a)\big),\text{OFF}\Big), & \textit{if } [\![a_i^0,a,\circ]\!] \in S; \\ \mu(a), & \textit{otherwise.} \end{cases} \tag{6}$$

The RFSG $M^A_{a^0_i}$ is called ***reconfiguration of*** M***, based on*** A***, after crossing*** a^0_i*.*

Let us see how this definition works in Figure 4 using the *arithmetic mean* as aggregation function, after crossing a sequence of zero-order arrows in Figure 3. After the arrow $a^0_1 = [xu]$ be crossed, Figure 4(a), the arrow $a^0_2 = [xy]$ is *updated* due to $a^1_1 = [\![[xu],[xy],\circ]\!]$ by the *arithmetic mean* between the fuzzy values $\mu_1(a^0_1), \mu_1(a^0_2)$ and $\mu_1(a^1_1)$, and by replacing the marker ON to OFF (the arrow a^0_2 becames inactive). In a second step and in the same manner, after the arrow $a^0_3 = [uy]$ be crossed, the arrow $a^1_1 = [\![[xu],[xy],\circ]\!]$ has its fuzzy value updated and becomes inactive, however, the arrow $a^0_5 = [vy]$ has only its fuzzy value updated since it is an active arrow targered by a connecting arrow (Figure 4(b)).



(a) (b)

Fig. 4. Reactivity of *RFSG* after crossing zero-order arrows $[xu]$ and $[xu][uy]$.

Proposition 1. *If A is a conjunctive (disjunctive) aggregation and $\left(\mu^A_{a^0_i}\right)_2(b) = \mu_2(b)$ for all $b \in S$, then $M^A_{a^0_i}$ is a subgraph (supergraph) of M.*

Proof. Given $b \in S$ and denoting $\big(\mu_1(a^0_i), \mu_1([\![a^0_i,b,\sigma]\!]), \mu_1(b)\big) = \overline{[\![a^0_i,b,\sigma]\!]}$:

– Case $[\![a^0_i,b,\sigma]\!] \in S$:

$$\left(\mu^A_{a^0_i}\right)_1(b) = A(\overline{[\![a^0_i,b,\sigma]\!]}) \leq \min(\overline{[\![a^0_i,b,\sigma]\!]}) \leq \mu_1(b).$$

– Case $[\![a^0_i,b,\sigma]\!] \notin S$:

$$\left(\mu^A_{a^0_i}\right)_1(b) = \mu_1(b)$$

.

Then $M^A_{a^0_i}$ is subgraph of M. The dual statement follows straightforwardly.

Proposition 2. *Let M' be a translation of M by $\lambda \in [-1, 1]$. If A is shift-invariant and $A(\bar{x}) + \lambda \in [0, 1]$ for all $\bar{x} \in [0, 1]^3$, then ${M'}^A_{a^0_i}$ is a translation of $M^A_{a^0_i}$ by λ.*

Proof. Given $b \in S$ and denoting $\big(\mu'_1(a^0_i), \mu'_1([\![a^0_i, b, \sigma]\!]), \mu'_1(b)\big) = \overline{[\![a^0_i, b, \sigma]\!]}$:

– Case $[\![a^0_i, b, \sigma]\!] \in S$:

$$\begin{aligned}\left({\mu'}^A_{a^0_i}\right)_1(b) &= A(\overline{[\![a^0_i, b, \sigma]\!]}) \\ &= A\big(\mu_1(a^0_i) + \lambda, \mu_1([\![a^0_i, b, \sigma]\!]) + \lambda, \mu_1(b) + \lambda\big) \\ &= A\left(\mu_1(a^0_i), \mu_1([\![a^0_i, b, \sigma]\!]), \mu_1(b)\right) + \lambda \\ &= \left(\mu^A_{a^0_i}\right)_1(b) + \lambda\end{aligned}$$

– Case $[\![a^0_i, b, \sigma]\!] \notin S$:

$$\left({\mu'}^A_{a^0_i}\right)_1(b) = \mu'_1(b) = \mu_1(b) + \lambda = \left(\mu^A_{a^0_i}\right)_1(b) + \lambda.$$

By hypoteses, $\mu_2(b) = \mu'_2(b)$, then $\left({\mu'}^A_{a^0_i}\right)_2(b) = \left(\mu^A_{a^0_i}\right)_2(b)$.

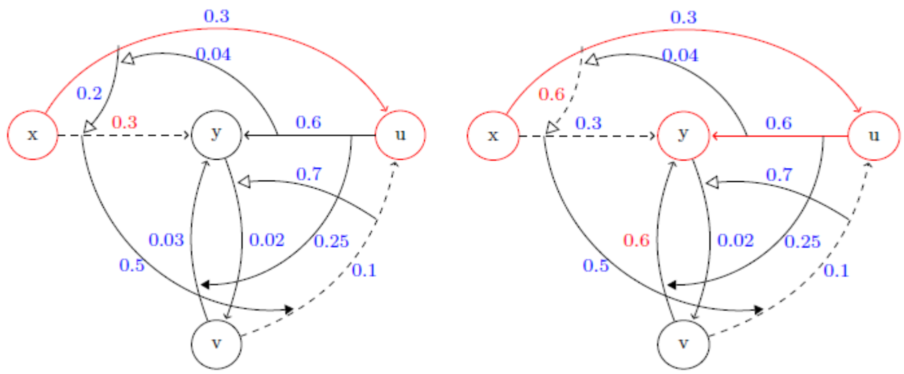
Next, we will provide an extension for the notion of reactivity presented in [13]. Just as it is done for the case of FRGs, each active zero-order arrow triggers an aggregation function.

Definition 19 (Reversal Fuzzy Reactive Graphs). *Consider M a $RFSG$, the sets $\Gamma = \{a^0_i \in S^0 : \mu_1(a^0_i) > 0\}$ of zero-order arrows and $A = \{A_1, ..., A_k : [0, 1]^3 \to [0, 1]\}$ of aggregation functions, and a function $A_g : \Gamma \to A$. The pair $M_R = \langle M, Ag\rangle$ is called* ***reversal fuzzy reactive graph (RFRG)****.*

If $a^0_i \in \Gamma$, the ***reconfiguration of*** *M_R* ***after crossing*** *a^0_i is the $RFRG$ $M^{a^0_i}_R = \langle M^{a^0_i}, A_g\rangle$, where $M^{a^0_i} = \langle W, \mu^{A_g}_{a^0_i}\rangle$ is a $RFSG$ s.t.*

$$\mu^{A_g}_{a^0_i}(b) = \begin{cases} \Big(A_g(a^0_i)\big(\mu_1(a^0_i), \mu_1([\![a^0_i, b, \bullet]\!]), \mu_1(b)\big), \text{ON}\Big), \; if\ [\![a^0_i, b, \bullet]\!] \in S; \\ \Big(A_g(a^0_i)\big(\mu_1(a^0_i), \mu_1([\![a^0_i, b, \circ]\!]), \mu_1(b)\big), \text{OFF}\Big), \; if\ [\![a^0_i, b, \circ]\!] \in S; \\ \mu(b), \; otherwise. \end{cases} \tag{7}$$

Example 12. Let M be the $RFSG$ at Figure 3, $\Gamma = \{[xy], [xu], [uy], [vy], [yv], [vu]\}$, $A = \{arith, max\}$, $A_g([xy]) = A_g([xu]) = A_g([yv]) = arith$ and $A_g([vy]) = A_g([uy]) = A_g([vu]) = max$. The Figure 5 contains $M^{[xu]}_R$ and $(M^{[xu]}_R)^{[uy]}$, respectively.



(a) $M_R^{[xu]}$ (b) $M_R^{[xu][uy]}$

Fig. 5. Reconfiguration of M_R.

3.2 Products of *RFSG*s

In the following, we will consider the *RFSG*s $M = \langle W, \mu : S \to [0,1] \times \{\text{ON}, \text{OFF}\}\rangle$ and $N = \langle V, \delta : T \to [0,1] \times \{\text{ON}, \text{OFF}\}\rangle$.

Definition 20 (Product of *RFSG***s).** *The* ***cartesian product*** *of the RFSGs M and N is the RFSG:* $M \times N = \langle W \times V\ ,\ \psi : (\{w\} \times T) \cup (S \times \{v\}) \to [0,1] \times \{\text{ON}, \text{OFF}\}\rangle$ *s.t.* $\psi(w,t) = \delta(t)$ *and* $\psi(s,v) = \mu(s)$.

Example 13. Consider M and N shown in Figure 6(a). The product $M \times N$ can be observed at Figure 6(b).

In order to define the product of *RFRG*s, we consider :

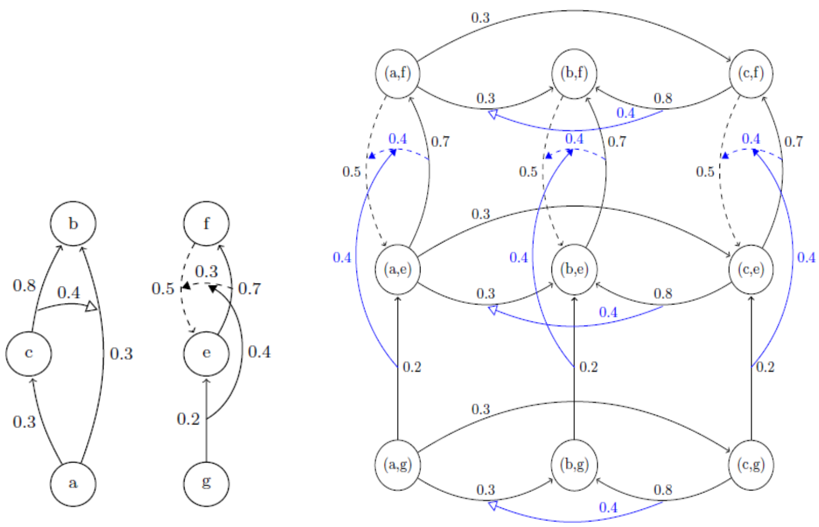
- The *RFRG*s $M_R = \langle M, Ag_M\rangle$ and $N_R = \langle N, Ag_N\rangle$;
- The functions $Ag_M : \Gamma_M \longrightarrow A_M$ and $Ag_N : \Gamma_N \longrightarrow A_N$;
- The sets of aggregations A_M and A_N;
- The sets of zero-order arrows Γ_M and Γ_N.

The aggregations $a_m \in A_M$ and $a_n \in A_N$ will be denoted by $(M, a_m) : [0,1]^3 \to [0,1]$ and $(N, a_n) : [0,1]^3 \to [0,1]$.

Definition 21 (Product de *RFRG***s).** *Consider the RFRGs* M_R *and* N_R*, the sets* $A_M \oplus A_N = \{(M, a_m) : a_m \in A_M\} \cup \{(N, a_n) : a_n \in A_N\}$ *and* $\Gamma_{M\times N} = \{\ a_i^0 \in (\{w\} \times T) \cup (S \times \{v\}) : \psi_1(a_i^0) > 0\}$*, and the function* $Ag_{M\times N} : \Gamma_{M\times N} \to A_M \oplus A_N$ *s.t.*

$$Ag_{M\times N}(a_i^0) = \begin{cases} \Big(N, Ag_N(t)\Big), & \text{if } a_i^0 = (w,t) \in \{w\} \times T; \\ \Big(M, Ag_M(s)\Big), & \text{if } a_i^0 = (s,v) \in S \times \{v\}. \end{cases} \tag{8}$$

The structure $M_R \times N_R = \langle M \times N, Ag_{M\times N}\rangle$ *is the* ***product of*** *RFRGs* M_R ***and*** N_R.



(a) *RFSGs* M and N (b) Cartesian product $M \times N$

Fig. 6.

The next proposition ensures that the updated product is obtained from the updated factors.

Proposition 3. *Consider the RFRGs* M_R, N_R, *the product* $M_R \times N_R$, $a_i^0 \in \Gamma_{M_R \times N_R}$ *and* $a \in (\{w\} \times T) \cup (S \times \{v\})$ *s.t.* $\big(\psi_1(a_i^0), \psi_1([\![a_i^0, a, \circ]\!]), \psi_1(a)\big) = \overline{[\![a_i^0, a, \circ]\!]}$ *and* $\big(\psi_1(a_i^0), \psi_1([\![a_i^0, a, \bullet]\!]), \psi_1(a)\big) = \overline{[\![a_i^0, a, \bullet]\!]}$. *Then,*

$$\psi_{a_i^0}^{Ag_{M\times N}}(a) = \begin{cases} \delta(t), & \text{if } C_1; \\ \mu(s), & \text{if } C_2; \\ \left(Ag_N(t)(\overline{[\![a_i^0, a, \circ]\!]}), \text{OFF}\right), & \text{if } C_3; \\ \left(Ag_N(t)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right), & \text{if } C_4; \\ \left(Ag_M(s)(\overline{[\![a_i^0, a, \circ]\!]}), \text{OFF}\right), & \text{if } C_5; \\ \left(Ag_M(s)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right), & \text{if } C_6; \end{cases} \tag{9}$$

For:

- $C_1 : a = (w, t)$ *and* $[\![a_i^0, a, \sigma]\!] \notin (\{w\} \times T) \cup (S \times \{v\})$;
- $C_2 : a = (s, v)$ *and* $[\![a_i^0, a, \sigma]\!] \notin (\{w\} \times T) \cup (S \times \{v\})$;
- $C_3 : a_i^0 = (w, t)$ *and* $[\![a_i^0, a, \circ]\!] \in (\{w\} \times T) \cup (S \times \{v\})$;
- $C_4 : a_i^0 = (w, t)$ *and* $[\![a_i^0, a, \bullet]\!] \in (\{w\} \times T) \cup (S \times \{v\})$;
- $C_5 : a_i^0 = (s, v)$ *and* $[\![a_i^0, a, \circ]\!] \in (\{w\} \times T) \cup (S \times \{v\})$;

– $C_6 : a_i^0 = (s, v)$ *and* $[\![a_i^0, a, \bullet]\!] \in (\{w\} \times T) \cup (S \times \{v\})$.

Proof. Indeed,

* Case $[\![a_i^0, a, \sigma]\!] \notin (\{w\} \times T) \cup (S \times \{v\})$,

– Case $a = (w, t)$: $\psi_{a_i^0}^{Ag_{M\times N}}(a) \overset{\text{def}}{=} \psi(a) \overset{\text{def}}{=} \delta(t)$.

– Case $a = (s, v)$: $\psi_{a_i^0}^{Ag_{M\times N}}(a) \overset{\text{def}}{=} \psi(a) \overset{\text{def}}{=} \mu(s)$.

* Case $[\![a_i^0, a, \circ]\!] \in (\{w\} \times T) \cup (S \times \{v\})$,

– Case $a = (w, t)$: $\psi_{a_i^0}^{Ag_{M\times N}}(a) \overset{\text{def}}{=} \left(Ag_{M\times N}(a_i^0)(\overline{[\![a_i^0, a, \circ]\!]}), \text{OFF}\right) \overset{\text{def}}{=} \left((N, Ag_N)(t)(\overline{[\![a_i^0, a, \circ]\!]}), \text{OFF}\right) \overset{\text{def}}{=} \left(Ag_N(t)(\overline{[\![a_i^0, a, \circ]\!]}), \text{OFF}\right)$.

– Case $a = (s, v)$: $\psi_{a_i^0}^{Ag_{M\times N}}(a) \overset{\text{def}}{=} \left(Ag_{M\times N}(a_i^0)(\overline{[\![a_i^0, a, \circ]\!]}), \text{OFF}\right) \overset{\text{def}}{=} \left((M, Ag_M)(s)(\overline{[\![a_i^0, a, \circ]\!]}), \text{OFF}\right) \overset{\text{def}}{=} \left(Ag_M(s)(\overline{[\![a_i^0, a, \circ]\!]}), \text{OFF}\right)$.

* Case $[\![a_i^0, a, \bullet]\!] \in (\{w\} \times T) \cup (S \times \{v\})$,

– Case $a = (w, t)$: $\psi_{a_i^0}^{Ag_{M\times N}}(a) \overset{\text{def}}{=} \left(Ag_{M\times N}(a_i^0)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right) \overset{\text{def}}{=} \left((N, Ag_N)(t)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right) \overset{\text{def}}{=} \left(Ag_N(t)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right)$.

– Case $a = (s, v)$: $\psi_{a_i^0}^{Ag_{M\times N}}(a) \overset{\text{def}}{=} \left(Ag_{M\times N}(a_i^0)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right) \overset{\text{def}}{=} \left((M, Ag_M)(s)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right) \overset{\text{def}}{=} \left(Ag_M(s)(\overline{[\![a_i^0, a, \bullet]\!]}), \text{ON}\right)$.

4 A Logic for *RFSG*s

In order to verify a system described by a *RFSG*, we provide a formal language and a fuzzy semantic. We use the set $S^{0^*} = \{a_i^0 \in S^0 \mid \mu_2(a_i^0) = \text{ON}\}$.

Definition 22 (Syntax). *Consider AtomProp a set of symbols (called atomic propositions) and* $p \in AtomProp$. *The set of formulas is generated by the following grammar:* $\varphi ::= p \mid true \mid false \mid (\neg\varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi) \mid (SNext(\varphi)) \mid (ANext(\varphi))$.

Given the formulas φ and ψ, we interpret:

$(\neg\varphi)$: φ is not true;
$(\varphi \wedge \psi)$: φ and ψ are true;
$(\varphi \vee \psi)$: φ or ψ is true;

$(\varphi \rightarrow \psi)$: If φ is true, then ψ is true;
$\varphi \leftrightarrow \psi$: φ is true if and only if ψ is true;
$SNext(\varphi)$: φ is true in some next state;
$ANext(\varphi)$: φ is true in all next states.

Definition 23. *A* ***model*** *over the set AtomProp is a pair* $\mathsf{M} = (M, V)$*, s.t.* $M = \langle W, \mu \rangle$ *is a RFSG and* $V : W \times AtomProp \rightarrow [0,1]$ *is a function called* ***fuzzy valuation****.*

Definition 24 (Semantics). *Consider* M *a model,* A *an aggregation function,* $\mathcal{F} = \langle [0,1], T, S, N, I, B, 0, 1 \rangle$ *a fuzzy semantics and* $w \in W$ *a state. The notation,* $[\![\mathsf{M}, w \models^A_{\mathcal{F}} \varphi]\!]$ *represents the* ***grade of uncertainty of a given formula*** φ***, at state*** w***, taking into account*** $\mathsf{M}, \mathcal{F}$ ***and*** A*. The value of* $[\![\mathsf{M}, w \models^A_{\mathcal{F}} \varphi]\!]$ *is defined in the following way:*

- $[\![\mathsf{M}, w \models^A_{\mathcal{F}} p]\!] = V(w,p)$, *for* $p \in AtomProp$.
- $[\![\mathcal{M}, w \models^A_{\mathcal{F}} true]\!] = 1$.
- $[\![\mathcal{M}, w \models^A_{\mathcal{F}} false]\!] = 0$.
- $[\![\mathcal{M}, w \models^A_{\mathcal{F}} (\varphi \wedge \psi)]\!] = \mathbf{T}([\![\mathcal{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \models^A_{\mathcal{F}} \psi]\!])$.
- $[\![\mathcal{M}, w \models^A_{\mathcal{F}} (\varphi \vee \psi)]\!] = \mathbf{S}([\![\mathcal{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \models^A_{\mathcal{F}} \psi]\!])$.
- $[\![\mathcal{M}, w \models^A_{\mathcal{F}} (\varphi \rightarrow \psi)]\!] = \mathbf{I}([\![\mathcal{M}, w \models^A_{\mathcal{F}} \varphi]\!], [\![\mathcal{M}, w \models^A_{\mathcal{F}} \psi]\!])$.
- $[\![\mathsf{M}, w \models^A_{\mathcal{F}} ANext(\varphi)]\!] = \underset{[ww'] \in S^{0*}}{\mathbf{T}} \left(\mathbf{I}\left(\mu([ww']), [\![\mathsf{M}^{[ww']}_A, w' \models^A_{\mathcal{F}} \varphi]\!]\right)\right)$ *since* $\mathsf{M}^{[ww']}_A$ *means* $\left(M^{[ww']}_A, V\right)$.
- $[\![\mathsf{M}, w \models^A_{\mathcal{F}} SNext(\varphi)]\!] = \underset{[ww'] \in S^{0*}}{\mathbf{S}} \left(\mathbf{T}\left(\mu([ww']), [\![\mathsf{M}^{[ww']}_A, w' \models^A_{\mathcal{F}} \varphi]\!]\right)\right)$.

The uncertainty degree that "$SNext(\varphi)$*" is true at the state* w *is computed by using the uncertainty degree that* φ *is true at some state with active relationship to* w*. On the other hand, the uncertainty degree that "*$ANext(\varphi)$*" is true at state* w *is computed by using the uncertainty degree that* φ *is true at every state with active relationship to* w*. We must highlight, contrary to what happens in the classic case, both* $SNext(\varphi)$ *and* $ANext(\varphi)$ *deal with all active edges in* S^{0*}*. The expression:* $[\![\mathsf{M}^{[w,w']}_A, w' \models \varphi]\!]$*, in this case, represents the uncertainty degree of the statement: "*φ *is true" at state* w' *after the active zero-order arrow* $a^0_i = [w, w']$ *has been crossed and the RFSG* M *has been updated to* $M^A_{a^0_i}$*.*

Observation: *Observe according to Notation 2 that the aplication* f *in the definition of* $ANext(\varphi)$ *is a fuzzy implication* $\mathbf{I}$*. Similarly, in the definition of* $SNext(\varphi)$*, the aplication* f *is a t-norm* $\mathbf{T}$*.*

Example 14. Consider the Figure 3 and take the atomic propositions: *High risk of contagion* and *Low risk of contagion*, according to the Table 1. What is the uncertainty degree at state x for the proposition: *"In some next state we have a low risk of contagion with a next state which has a higher risk of contagion?"* The assertion can be expressed as: $SNext(L \wedge SNext(H))$.

Table 1. Truth values of propositions on each state.

	x	y	u	v
H	0.2	0.8	0.3	0.01
L	0.1	0.9	0.15	0.2

Assuming the *arithmetical mean* as the unique aggregation function, the *Gödel Semantic* $\mathcal{F}_G$ and $\varphi = L \wedge SNext(H)$,

$$[\![\mathsf{M}, x \models^A_{\mathcal{F}_G} SNext(\varphi)]\!] \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M\big(0.3, [\![\mathsf{M}^{[xu]}_A, u \models^A_{\mathcal{F}_G} \varphi]\!]\big),$$

$$\mathbf{T}_M\big(0.4, [\![\mathsf{M}^{[xy]}_A, y \models^A_{\mathcal{F}_G} \varphi]\!]\big)\Big) \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M(0.3, 0.15), \mathbf{T}_M(0.4, 0.01)\Big) = 0.15$$

Since,

a) $[\![\mathsf{M}^{[xu]}_A, u \models^A_{\mathcal{F}_G} \varphi]\!] \stackrel{\text{def}}{=} \mathbf{T}_M\Big([\![\mathsf{M}^{[xu]}_A, u \models^A_{\mathcal{F}_G} SNext(H)]\!], [\![\mathsf{M}^{[xu]}_A, u \models^A_{\mathcal{F}_G} L]\!]\Big) =$ $\mathbf{T}_M(0.6, 0.15) = 0.15$ due to $[\![\mathsf{M}^{[xu]}_A, u \models^A_{\mathcal{F}_G} SNext(H)]\!] \stackrel{\text{def}}{=}$ $\mathbf{S}_M\Big(\mathbf{T}_M\big(\mu^A_{[xu]}([uy]), [\![\mathsf{M}^{[xu][uy]}_A, y \models^A_{\mathcal{F}_G} H]\!]\big)\Big) = \mathbf{S}_M\Big(\mathbf{T}_M\big(0.6, 0.8\big)\Big) = 0.6;$

b) $[\![\mathsf{M}^{[xy]}_A, y \models^A_{\mathcal{F}_G} \varphi]\!] \stackrel{\text{def}}{=} \mathbf{T}_M\Big([\![\mathsf{M}^{[xy]}_A, y \models^A_{\mathcal{F}_G} SNext(H)]\!], [\![\mathsf{M}^{[xy]}_A, y \models^A_{\mathcal{F}_G} L]\!]\Big) =$ $\mathbf{T}_M(0.01, 0.9) = 0.01.$ due to $[\![\mathsf{M}^{[xy]}_A, y \models^A_{\mathcal{F}_G} SNext(H)]\!] \stackrel{\text{def}}{=}$ $\mathbf{S}_M\Big(\mathbf{T}_M\big(\mu^A_{[xy]}([yv]), [\![\mathsf{M}^{[xy][yv]}_A, v \models^A_{\mathcal{F}_G} H]\!]\big)\Big) = \mathbf{S}_M\Big(\mathbf{T}_M\big(0.02, 0.01\big)\Big) =$ 0.01

In this case, in order to calculate the uncertainty degree at state v for the same proposition, we should note that the state v has only one next state y (the inactive arrow $[vu]$ is not considered). Therefore,

$$[\![\mathsf{M}, v \models^A_{\mathcal{F}_G} SNext(\varphi)]\!] \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M\big(0.03, [\![\mathsf{M}^{[vy]}_A, y \models^A_{\mathcal{F}_G} \varphi]\!]\big)\Big) = 0.01.$$

since $[\![\mathsf{M}^{[vy]}_A, y \models^A_{\mathcal{F}_G} \varphi]\!] \stackrel{\text{def}}{=} \mathbf{T}_M\Big([\![\mathsf{M}^{[vy]}_A, y \models^A_{\mathcal{F}_G} SNext(H)]\!], [\![\mathsf{M}^{[vy]}_A, y \models^A_{\mathcal{F}_G} L]\!]\Big) =$ $\mathbf{T}_M(0.01, 0.9) = 0.01$ due to $[\![\mathsf{M}^{[vy]}_A, y \models^A_{\mathcal{F}_G} SNext(H)]\!] \stackrel{\text{def}}{=} \mathbf{S}_M\Big(\mathbf{T}_M\big(\mu^A_{[vy]}([yv]), [\![\mathsf{M}^{[vy][yv]}_A, v \models^A_{\mathcal{F}_G} H]\!]\big)\Big) =$ $\mathbf{S}_M\Big(\mathbf{T}_M\big(0.02, 0.01\big)\Big) = 0.01.$

5 Modeling a Tank Level Control System

In industrial processes that use tanks, the control of the fluid level is a common practice. Even with a relatively simple structure, logic controllers are often used. The study and the modeling of tank plants and logic controllers are important because they provide the understanding of the current scenario of the system, causing benefits such as: the increase of productivity and the prevention of accidents [5].

The Figure 7(a) shows a scheme where a tank control system is built with three signal transmitters $\{ST_1, ST_2, ST_3\}$, two pumps $\{P_1, P_2\}$ and a channel for fluid inlet called $START$. The dynamics of the system works as follows:

- **Fluid level rising**: At $START$ the fluid starts to be inserted into the tank while pumps P_1 and P_2 are on standby receiving a minimum electric current. When the fluid level triggers ST_2, P_1 receives an increment of electric current and is activated. If the fluid level continues to rise and trigger ST_3, the pump P_2 receives an increment of electric current and is also actived.
- **Fluid level decreasing**: When the fluid level is maximum, the pumps P_1 and P_2 are active. When the fluid level decreases, the ST_3 is triggered and P_2 goes to standby with a decrease in its electric current. If the fluid level continues to decrease, ST_2 is triggered and P_1 goes to standby with a decreasing in its electric current.

The signal transmitter receives the difference pressure of two points with different weights and converts it into a proportional electrical signal. This electric signal is sent to pumps [5].

Consider in Figure 7(b):

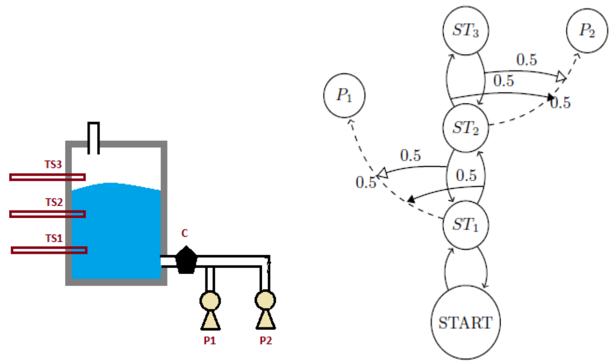
- The set of arrows S;
- The set of worlds $W = \{ST_1, ST_2, ST_3, P_1, P_2, START\}$;
- The membership function $\mu : S \to [0, 1] \times \{\text{ON}, \text{OFF}\}$ which assign to each arrow in S, the electric signal generated when they are crossing;
- The function $A_g : S^0 \to A$, where $A = \{T_L, S_L\}$.

The $RFRG$ $M_R = \langle M, A_g \rangle$ models the system of tank control above. These systems could also be model by using a FSG in which all arrows are active and all high-order arrows are connecting. However, in this case, there would be no possibility of working on deactivation of the pumps.

The reconfiguration of M_R, after crossing the arrows sequence $[ST_1\ ST_2]$, $[ST_1\ ST_2][ST_2\ ST_3]$ and $[ST_1\ ST_2][ST_2\ ST_3][ST_3\ ST_2]$, can be observed in Figure 8. Assuming:

- $\mu(a_i^0) = 1$, for all $a_i^0 \in S^0 - \{[ST_1\ P_1], [ST_2\ P_2]\}$;
- $A_g([ST_1\ ST_2]) = A_g([ST_2\ ST_3]) = S_L$;
- $A_g([ST_2\ ST_1]) = A_g([ST_3\ ST_2]) = T_L$.

The fuzzy value and the status of the arrow $[ST_1\ P_1]$ after the arrow $[ST_1\ ST_2]$ has been crossed is calculate in the follow way:



(a) Tank Scheme (b) *RFRG* M_R

Fig. 7. Model of tank tontrol system.

$$\begin{aligned}\mu^{Ag}_{[ST_1\ ST_2]}([ST_1\ ST_2]) &= \Big(S_L(1, 0.5, 0.5), \text{ON}\Big)\\ &= \Big(S_L(1, S_L(0.5, 0.5)), \text{ON}\Big)\\ &= \Big(S_L(1, 1), \text{ON}\Big)\\ &= \Big(1, \text{ON}\Big)\end{aligned}$$

Consider the propositions

p: "P_1 *is active*" and q:"P_2 *is active*"

for the model $\mathsf{M} = \langle M, V\rangle$, with $V(ST_1, p) = 0.05$, $V(ST_2, p) = 0.08$, $V(ST_3, p) = 0.6$, $V(ST_1, q) = 0.01$, $V(ST_2, q) = 0.5$ and $V(ST_3, q) = 0.7$. Using the *Gödel Fuzzy Semantics*, we are able to compute the true value of the formula $SNext(p\wedge q)$ to the states ST_2 and ST_3:

$[\![\mathsf{M}, ST_2 \models^{A_g}_{\mathcal{F}_G} SNext(p \wedge q)]\!] = \mathbf{S}_M\Big(\mathbf{T}_M\big(0.1, [\![\mathsf{M}^{[ST_2\ ST_1]}_{A_g}, ST_1 \models^{A_g}_{\mathcal{F}_G} (p \wedge q)]\!]\big),$

$\mathbf{T}_M\big(0.1, [\![\mathsf{M}^{[ST_2\ ST_3]}_{A_G}, ST_3 \models^{A_G}_{\mathcal{F}_G} (p \wedge q)]\!]\big)\Big) = 0.1$ and

$[\![\mathsf{M}, ST_3 \models^{A_g}_{\mathcal{F}_G} SNext(p\wedge q)]\!] = \mathbf{S}_M\Big(\mathbf{T}_M\big(0.1, [\![\mathsf{M}^{[ST_3\ ST_2]}_{A_g}, ST_2 \models^{A_g}_{\mathcal{F}_G} (p\wedge q)]\!]\big)\Big) = 0.08$.

So, the degree of the sentence,

"*There is a next state in which the pumps P1 and P2 are working*"

at the state ST_2 is 0.1 and at the state ST_3 is 0.08.

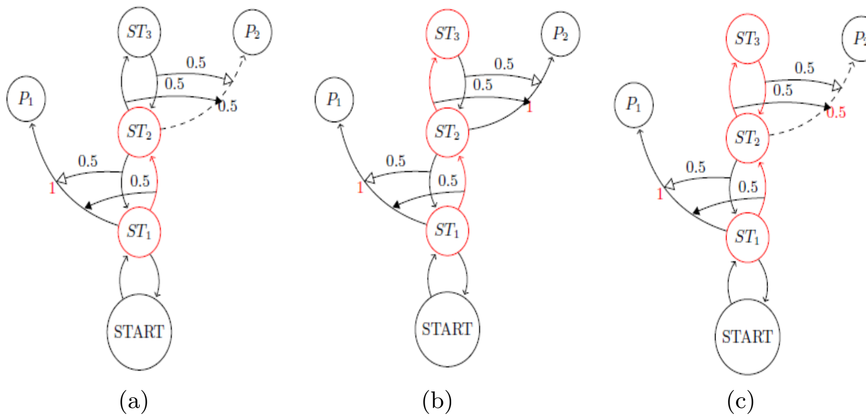


Fig. 8. M_R configuration after: (a) $[ST_1\ ST_2]$, (b) $[ST_1\ ST_2][ST_2\ ST_3]$ and (c) $[ST_1\ ST_2][ST_2\ ST_3][ST_3\ ST_2]$.

6 Final Remarks

This paper introduces a structure called *Reversal Fuzzy Switch Graph* ($RFSG$) which extends the concept of Fuzzy Switch Graph (FSG) introduced by Santiago et al.(2020) [13] by assigning the device to enable/disable arrows. We present the cartesian product of $RFSG$s and produce an application to demonstrate how RFRGs can model dynamic systems. The model is accompanied with a formal logic which enables the verification of properties.

In order to keep the text reduced, further algebraic operators and concepts, like simulation and bisimulations, were not exposed. These constructions as well as the relationship of $RFSG$s with other models that describe reactive systems, other types of logic and other notions of certainty (such as fuzzy interval logic), will be subject of future works.

7 Acknowledgements

Santiago was supported by National Council for Scientific and Technological Development (CNPq) within the project 312053/2018-5, by Coordination for the Improvement of Higher Education Personnel (CAPES) within the project Capes-Print 88887.363001/2019-00.

Martins and Figueiredo were supported by ERDF, through the COMPETE 2020 Program, and by FCT, within the projects PTDC/CCI-COM/30947/2017, UIDB/04106/2020 and UIDP/04106/2020.

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