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Graded epistemic logic with public announcement ☆

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Work dedicated to **Prof. José Manuel Valença** on the occasion of his Festschrift. The authors sincerely recognize the inestimable impact of his carrier on the affirmation of the Portuguese Software Engineering.

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A B S T R A C T

This work introduces a new fuzzy epistemic logic with public announcement with fuzzyness on both transitions and propositions. The interpretation of the connectives is done over the Gödel algebra and the interpretation of public announcements in this logic generalises the traditional update one. The core idea is that, the effect of a public announcement is reflected on the transitions degrees of the models. The update takes in account not only the truth degree of the announcement, at a target state, but also the degree of the transitions reaching that state. We prove the soundness of all axioms of the multi-agent epistemic logic with public announcements with respect to this graded semantics. Finally, we introduce the notion of bisimulation and prove the modal invariance property for our logic.

1. Introduction

Rooted on works of Hintikka [10], the development of new modal logics to reason about knowledge in a multi-agent systems framework [7] have been raised the attention of the modern logic and computer science communities. These logics of knowledge describe how an agent reasons about his own knowledge and about the knowledge of other agents. We say that an agent knows a fact φ if φ is true in every state that the agent considers possible. "*The intuition is that if an agent does not have complete knowledge about the world, he will consider a number of possible worlds. These are his candidates for the way the world actually is*" [7]. The analysis and design of autonomous cooperative systems or secure communication protocols are just two application scenarios where these logics can be useful. When thinking, for instance, on security key sharing protocols, in games strategy design or even on the design of autonomous collaborative systems as discovering robot teams, the existence of mechanisms to update knowledge models with announced information is crucial. This is the base principle of epistemic logics with public announcements, which are logics to model and reasoning about epistemic action, i.e., actions that change agents or group of agents knowledge and beliefs.

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Situations where uncertainty is present in the levels of agent knowledge and beliefs is widely common in practical scenarios. For instance, it is not unusual to believe in some fact with some degree of possibility. In situations like *Ana believes that her father has a strong preference for Bob, which means that she believes that he will give a sweet to Bob rather than to Clara. For instance, in the interval* [0, 1], *we may say that her belief is* 0.9. In this case the belief is not true or false. In more practical scenarios, as in cooperative robotic exploring missions, the imprecision in the perception of the sensors can induce situations where the presence of an obstacle would be better expressed as a graded assertion than to a bivalent one. In this work we deal with graded knowledge: both relations and atomic propositions are graded.

This work introduces a new fuzzy epistemic logic with public announcement. The proposal results from the combination of previous works of the authors. On the one hand, it inherits graded transitions from the epistemic logics parametrically build in the method of [2]. On the other hand, the propositions have graded semantics, accordingly to the fuzzy logics with structured states presented in [15]. The interpretation of the fuzzy connectives is done over the Gödel algebra (cf. [8]).

Against of what happens with other (graded) approaches in the literature (e.g. [21,3,12]), the public announcements introduced here generalises the standard interpretation of public announcements. The main difference is the models update principle adopted. The core idea is that the effect of a public announcement is reflected in transitions degrees of the models. The update takes in account not only the truth degree of the announcement in a target state, but also the degree of the transitions reaching that state. Hence, we capture the standard update construction that erases the states where the announcement is false: when consider the standard (crisp) case, our construction assigns to the transition to the state where the announce if false the value 0, turning it unreachable (and thus, semantically irrelevant) from the state where the sentence is being evaluated.

Finally we introduced a notion of bisimulation for which the modal invariance property is proved. This result provides the basis for a number of models reduction techniques, essential for model checking analysis in complex multi-agent scenarios involving knowledge. The paper is illustrated with two running examples, including a variant of the popular example of the three cards game from [23].

Outline. The remaining of the paper is organized as follows. Section 2 recall the standard multi-agent epistemic logic with public announcement. Section 3, represents the core of the paper: it introduces a Fuzzy Dynamic Epistemic Logic with Public announcements; then it studies the validity of the axioms of the standard epistemic logic with public announcements; finally, we illustrate the logic with two representative examples. Section 4 introduces a notion of bisimulation for the logic and shows that it enjoys the modal invariance property. Finally, Section 5, discusses some lines of related and future work.

2. Preliminaries: epistemic logic with public announcement

Multi-agent epistemic logic has been investigated in Computer Science [7] to represent and reason about agents or groups of agents knowledge and beliefs. This section recalls the basis of $\mathcal{ELP}$, the Multi-agent epistemic logic with public announcements.

The epistemic language consists of a pair (Prop, A), where Prop is a set of countably many proposition symbols and A is a finite set of agents. The set of (Prop, A)-formulas for the epistemic logic with public announcements, $\text{Fm}^{\mathcal{ELP}}(\text{Prop}, A)$ in symbols, is defined by the following grammar:

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\varphi]\varphi$$

where $p \in \text{Prop}$ and $a \in A$.

The other standard connectives can be defined as abbreviations, namely $\bot \equiv \neg\top$, $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi \equiv \neg(\varphi \wedge \neg\psi)$, $B_a\varphi \equiv \neg K_a \neg\varphi$ and $\langle\chi\rangle\varphi \equiv \neg[\chi]\neg\varphi$.

Definition 1. A (Prop, A)-*multi-agent model* is a tuple $\mathcal{M} = (W, R, V)$ where

- W is a non-empty set of states;
- $R = (R_a \subseteq W \times W)_{a \in A}$ is an A-family of binary relations over W;
- $V : \text{Prop} \to \mathcal{P}(W)$ is a valuation function.

A (Prop, A)-multi-agent model is an *epistemic model*, if the relations R_a, $a \in A$, are equivalence relations.

Definition 2. Given a (Prop, A)-multi-agent model $\mathcal{M}$. The notion of satisfaction $\mathcal{M}, w \models^{\mathcal{ELP}} \varphi$ is defined as follows

- $\mathcal{M}, w \models^{\mathcal{ELP}} p$ iff $w \in V(p)$
- $\mathcal{M}, w \models^{\mathcal{ELP}} \bot$ is false
- $\mathcal{M}, w \models^{\mathcal{ELP}} \neg\varphi$ iff $\mathcal{M}, w \not\models^{\mathcal{ELP}} \varphi$
- $\mathcal{M}, w \models^{\mathcal{ELP}} \varphi \wedge \psi$ iff $\mathcal{M}, w \models^{\mathcal{ELP}} \varphi$ and $\mathcal{M}, w \models^{\mathcal{ELP}} \psi$
- $\mathcal{M}, w \models^{\mathcal{ELP}} K_a\varphi$ iff for all $w' \in W$, wR_aw' implies $\mathcal{M}, w' \models^{\mathcal{ELP}} \varphi$
- $\mathcal{M}, w \models^{\mathcal{ELP}} [\chi]\varphi$ iff $\mathcal{M}, w \models^{\mathcal{ELP}} \chi$ implies $\mathcal{M}|_\chi, w \models^{\mathcal{ELP}} \varphi$

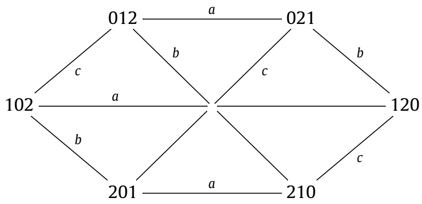

Fig. 1. Model $\mathcal{M}$ of **a**na, **b**ob and **c**lara beliefs.

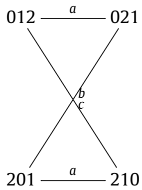

Fig. 2. Representation of $M|_{\neg 1_a}$.

where $\mathcal{M}|_\chi = (W|_\chi, R|_\chi, V|_\chi)$ is the (Prop, A)-multi-agent model defined as follows:

- $W|_\chi = \{w \in W \mid \mathcal{M}, w \models \chi\}$;
- for any $a \in \mathcal{A}$, $(R|_\chi)_a = R_a \cap (W|_\chi \times W|_\chi)$;
- for any $p \in \text{Prop}$, $V|_\chi(p) = V(p) \cap W|_\chi$.

In the following examples we adopt the usual representation of epistemic models where the reflexive loops are omitted in the diagrams.

Example 1 *(An adaptation from [23]).* Suppose a father has three envelopes, each containing: **0**, **1** and **2** dollars inside respectively. The father has three children: **a**na, **b**ob and **c**lara. Each child receives one envelope and do not know content of the envelopes of the other children.

We use proposition symbols y_x, where $y \in \{0, 1, 2\}$ and $x \in \{a, b, c\}$ to express "child x has envelope 0, 1 and 2". We name each state by the envelope that each child has in that state, for instance 012 is the state where child **a** has **0**, child **b** has **1** and child **c** has **2**. The epistemic model $M = (W, R, V)$, where

- $W = \{012, 021, 102, 120, 201, 210\}$
- $R_a = \{(012, 012), (012, 021), (021, 021), \ldots\}$, $R_b = \{\ldots\}$...
- $V(0_a) = \{012, 021\}$, $V(1_a) =$
  $\{102, 120\}$,...

represents the epistemic state of each agent. Its diagramatic presentation is done in Fig. 1.

It is not difficult to see, for instance, that $012 \models^{\mathcal{ELP}} B_b 0_a$ and $012 \models^{\mathcal{ELP}} B_a K_c 2_c$ hold. We are now going to make a public announcement and evaluate the knowledge of an agent after that. Consider the announcement "agent a does not have 1 dollar in the envelope" ($\neg 1_a$). We want to verify if agent c knows if agent a has 0 dollars ($K_c 0_a$). This can be expressed as:

$$M, 012 \models^{\mathcal{ELP}} [\neg 1_a] K_c 0_a \tag{1}$$

The epistemic model $M|_{\neg 1_a}$ that represents the epistemic state of each agent after the announcement is the represented in Fig. 2.

Now, we can check if $K_c 0_a$ holds in $M|_{\neg 1_a}$

$$M|_{\neg 1_a}, 012 \models^{\mathcal{ELP}} K_c 0_a \tag{2}$$

Now, we have to check if 0_a is true in every state connected to state 012 via the relation R_c. Since there are no other states other than 012 itself, we check only this state, yielding

$$M|_{\neg 1_a}, 012 \models^{\mathcal{ELP}} 0_a \tag{3}$$

This evaluates to true since $012 \in V(0_a)$, and therefore $M|_{\neg 1_a}, 012 \models^{\mathcal{ELP}} K_c 0_a$ is also true.

Axiomatization

The axiomatization of $\mathcal{ELP}$ is an extension of the axiomatization of multi-agent epistemic logic with the public announcement axioms.

Axioms

A) *All instantiations of propositional tautologies*,
B) $K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$,
C) $K_a\varphi \to \varphi$,
D) $K_a\varphi \to K_aK_a\varphi$,
E) $\varphi \to K_aB_a\varphi$
F) $[\varphi]p \leftrightarrow (\varphi \to p)$
G) $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi])$
H) $[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$
I) $[\varphi][\psi]\chi \leftrightarrow \left[\varphi \wedge [\varphi]\psi\right]\chi$
J) $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\neg\psi)$

Inference rules

From φ, infer $[\chi]\varphi$
From φ, infer $K_a\varphi$

This axiomatization is sound and complete with respect to the semantics presented above. The proof of completeness is based on the reduction of epistemic logic with public announcements to the standard multi-agent epistemic logic, since any formula of $\mathcal{ELP}$ is equivalent to a formula of multi-agent epistemic logic. Hence, completeness follows from the completeness of multi-agent epistemic logic [23].

There are other equivalent presentations of this logic in the literature. For instance, it is usual to consider the axiom $\neg K_a\varphi \to K_a\neg K_a\varphi$ on place of $\varphi \to K_aB_a\varphi$. The axiomatic adopted here is more convenient to work in our fuzzy semantics.

3. Fuzzy epistemic logic with public announcement

In the first part of this section we present the language and semantics of our fuzzy epistemic logic with public announcements $\mathcal{FELP}$ illustrating it with some examples. Then, we prove that all axioms of Public Announcement Logic presented above are valid in our semantics.

3.1. The logic

The signatures of the fuzzy epistemic logic with public announcements are the same as for $\mathcal{ELP}$. The set of formulas is defined as in $\mathcal{ELP}$. However, there are some connectives inter-definable in $\mathcal{ELP}$ that can be considered as primitives in our logic. In fact, as we will see later, in the semantics adopted, there is no redundancy in this choice. For example, the Morgan laws does not hold in the Gödel algebra that paves our semantics; hence, for instance, we can not define the conjunction from the disjunction.

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid K_a\varphi \mid B_a\varphi \mid [\varphi]\varphi \mid \langle\varphi\rangle\varphi$$

As usual we use $\perp$ to denote $\neg\top$ and $\varphi \leftrightarrow \psi$ to abbreviate $(\varphi \to \psi) \wedge (\psi \to \psi)$.

Definition 3. A (Prop, A)-*model* is a tuple $\mathcal{M} = (W, R, V)$ where

- W is a non-empty set of states;
- $R = (R_a : W \times W \to [0, 1])_{a\in A}$ is an A-family of fuzzy relations;
- V : Prop $\times\, W \to [0, 1]$ is a valuation function

A (Prop, A)-model M is a (Prop, A)-*multi-agent Fuzzy Epistemic Model* if

- R is *graded-reflexive*, i.e. for any $a \in A$,

$$\text{for any } w \in W,\ R_a(w, w) = 1 \tag{4}$$

- R is *graded-symmetric*, i.e. for any $a \in A$,

$$\text{for any } w, w' \in W,\ R_a(w, w') = R_a(w', w) \tag{5}$$

- and R is *graded-transitive*, i.e. for any $a \in A$,

$$\text{for any } w, w', w'' \in W,\ R_a(w, w'') \geq \min\left\{R_a(w, w'), R_a(w', w'')\right\} \tag{6}$$

The interpretation of the fuzzy connectives of our logic is based on Gödel algebra. The *implication operator* is the function $I : [0,1] \to [0,1]$ defined by

$$I(x, y) = \begin{cases} 1 & x \leq y \\ y & \text{otherwise} \end{cases} \tag{7}$$

and the *negation operator* is the function $N : [0,1] \to [0,1]$ defined by

$$N(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Finally, the operators of conjunction and disjunction are the numeric maximum and minimum operations in [0, 1]. We use $\bigwedge_{i\in I} a_i$ and $\bigvee_{i\in I} a_i$ to denote the iterated application of the binary operations max and min to a set. Next lemma recalls some well known properties which hold in the Gödel algebra. A more carefully treatment of the issue, including its proofs can be found in [20].

Lemma 1. *Are true the following properties:*

$$I(a, b) = 1 \quad \textit{iff} \quad a \leq b \tag{9}$$

$$I(a, I(b, c)) = I(\min\{a, b\}, c) \tag{10}$$

$$\min\{a, I(a, b)\} = \min\{a, b\} \tag{11}$$

$$I(a, \bigwedge_{i\in J}\{b_i\}) = \bigwedge_{i\in J}\{I(a, b_i)\} \tag{12}$$

$$\bigwedge_{i\in J}\{I(a_i, b)\} = I\big(\bigvee_{i\in J}\{a_i\}, b\big) \tag{13}$$

$$a \leq b \quad \textit{implies} \quad I(c, a) \leq I(c, b) \tag{14}$$

$$a \leq b \quad \textit{implies} \quad I(b, c) \leq I(a, c) \tag{15}$$

$$\min(a, \bigvee_{i\in J} a_i) = \bigvee_{i\in J} \min(a, a_i) \tag{16}$$

Definition 4. Let (Prop, A) be a signature, $\mathcal{M} = (W, R, V)$ be a (Prop, A)-model and $\varphi \in \text{Fm}^{\mathcal{FELP}}(\text{Prop}, A)$ a formula. The *graded satisfaction relation for* $\mathcal{FELP}$

$$\models : W \times \text{Fm}^{\mathcal{FELP}}(\text{Prop}, A) \to [0,1]$$

is the fuzzy relation defined as follows:

- $(\mathcal{M}, w \models \top) = 1$
- $(\mathcal{M}, w \models p) = V(p, w)$
- $(\mathcal{M}, w \models \neg\varphi) = N(\mathcal{M}, w \models \varphi)$
- $(\mathcal{M}, w \models \varphi \wedge \psi) = \min\big\{(\mathcal{M}, w \models \varphi), (\mathcal{M}, w \models \psi)\big\}$
- $(\mathcal{M}, w \models \varphi \to \psi) = I\big(\mathcal{M}, w \models \varphi), (\mathcal{M}, w \models \psi)\big)$
- $(\mathcal{M}, w \models \varphi \vee \psi = \max\big\{(\mathcal{M}, w \models \varphi), (\mathcal{M}, w \models \psi)\big\}$
- $(\mathcal{M}, w \models K_a\varphi) = \bigwedge_{w'\in W} \big\{I(R_a(w, w'), (M, w' \models \varphi))\big\}$
- $(\mathcal{M}, w \models B_a\varphi) = \bigvee_{w'\in W} \big\{\min\{R_a(w, w'), (M, w' \models \varphi)\}\big\}$
- $(\mathcal{M}, w \models [\chi]\varphi) = I\big((\mathcal{M}, w \models \chi), (\mathcal{M}|_\chi, w \models \varphi)\big)$

where the (Prop, A)-model $\mathcal{M}|_\chi$ is the tuple $\mathcal{M}|_\chi = (W, R|_\chi, V)$ such that, for any $a \in A$, $w, w' \in W$, $(R|_\chi)_a(w, w') = \min\{R_a(w, w'), (M, w' \models \chi)\}$. As usual, we say that a formula φ *is true in* M if for any state $w \in W$, $(M, w \models \varphi) = 1$; and that φ *is valid* if true in all the (Prop, A)-models.

Now we present two examples in order to illustrate our approach. First, we look at a variation of Example 1. Then, we discuss an example about two robots that control the flow and the level of an water reservoir.

Example 2 *(Continuation of Example 1).* Suppose now that the children are not allowed to open the envelopes, but they are curious enough to try to see through the envelope. If the envelope is empty they can see through it and they are almost

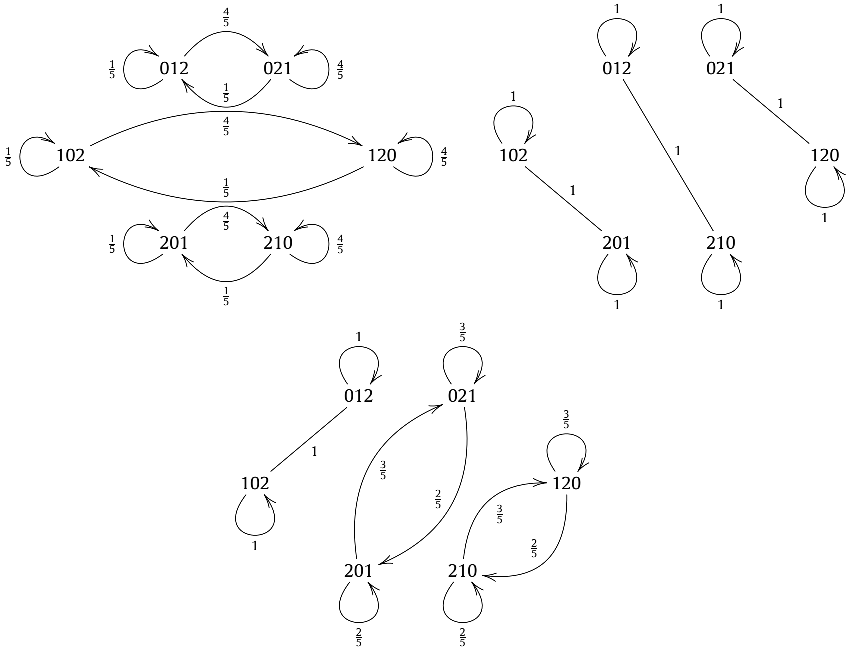

Fig. 3. Model $\mathcal{H}$: **a**na's, **b**ob's and clara's beliefs.

sure that the envelope has **0**. Moreover, suppose that the one dollar bill is darker than 2 dollar bill and it has a greater possibility to be guessed by the children. So, in each state the children assign the following propositional values, according to the valuation V depicted in the following tables:

	$\mathbf{0_a}$	$\mathbf{1_a}$	$\mathbf{2_a}$
012	$\frac{9}{10}$	$\frac{1}{10}$	$\frac{2}{10}$
021	$\frac{9}{10}$	$\frac{1}{10}$	$\frac{2}{10}$
102	$\frac{1}{10}$	$\frac{7}{10}$	$\frac{3}{10}$
120	$\frac{1}{10}$	$\frac{7}{10}$	$\frac{3}{10}$
201	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{8}{10}$
210	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{8}{10}$

	$\mathbf{0_b}$	$\mathbf{1_b}$	$\mathbf{2_b}$
012	$\frac{2}{10}$	$\frac{8}{10}$	$\frac{2}{10}$
021	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{6}{10}$
102	$\frac{7}{10}$	$\frac{2}{10}$	$\frac{1}{10}$
120	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{6}{10}$
201	$\frac{7}{10}$	$\frac{2}{10}$	$\frac{1}{10}$
210	$\frac{2}{10}$	$\frac{8}{10}$	$\frac{2}{10}$

	$\mathbf{0_c}$	$\mathbf{1_c}$	$\mathbf{2_c}$
012	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{9}{10}$
021	$\frac{2}{10}$	$\frac{7}{10}$	$\frac{2}{10}$
102	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{7}{10}$
120	$\frac{9}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
201	$\frac{1}{10}$	$\frac{8}{10}$	$\frac{2}{10}$
210	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{8}{10}$

Suppose now that the children have the following beliefs:

1. **a**na believes that the father has a strong preference for **b**ob, which means that she believes that he will give the envelope with higher value to **b**ob than to **c**lara. Her belief is $\frac{4}{5}$; moreover her belief that the value is less is $\frac{1}{5}$
2. **c**lara also believes that the father has a preference for **b**ob. Her belief is $\frac{3}{5}$. But if she has the envelope **2** then she believes that the father has no preference between **a**na and **b**ob. In such case her belief is 1.
3. **b**ob does not believe that the father has any preference between **a**na and **c**lara. So, his beliefs are all 1.

The draws in Fig. 3 represent the beliefs of **a**na, **b**ob and **c**lara. We draw them separately for clarity sake.

Suppose the current state is 012 and **a**na announces that she has either 0 or 2 dollars. We want to evaluate the knowledge of **c**lara about **a**na having $\mathbf{0_a}$ dollars. This can be expressed by the formula $[\mathbf{0_a} \vee \mathbf{2_a}]K_c\mathbf{0_a}$. In order to evaluate this formula at 012 we have to obtain the new model $\mathcal{H}|_{(\mathbf{0_a} \vee \mathbf{2_a})}$ as in Fig. 4. We only draw **c**lara's belief once the formula only involves her beliefs.

$$\begin{aligned}\mathcal{H}|_{(\mathbf{0_a}\vee\mathbf{2_a})}, 012 \models K_c 0_a = & \bigwedge\left\{I(R_c(012,012), (\mathcal{H}|_{(\mathbf{0_a}\vee\mathbf{2_a})}, 012 \models 0_a)), I(R_c(012,102), (\mathcal{H}|_{(\mathbf{0_a}\vee\mathbf{2_a})}, 102 \models 0_a))\right\} \\ = & \bigwedge\left\{I(\tfrac{9}{10}, \tfrac{9}{10}), I(\tfrac{3}{10}, \tfrac{1}{10})\right\} = \bigwedge\left\{1, \tfrac{1}{10}\right\} = \tfrac{1}{10}\end{aligned}$$

We are ready to evaluate $\mathcal{H}, 012 \models [\mathbf{0_a} \vee \mathbf{2_a}]K_c\mathbf{0_a}$

$$\begin{aligned}\mathcal{H}, 012 \models [\mathbf{0_a} \vee \mathbf{2_a}]K_c\mathbf{0_a} & = I\big((\mathcal{H}, 012 \models \mathbf{0_a} \vee \mathbf{2_a}), (\mathcal{H}|_{(\mathbf{0_a}\vee\mathbf{2_a})}, 012 \models K_c\mathbf{0_a})\big) \\ & = I\big(\tfrac{9}{10}, \tfrac{1}{10}\big) = \tfrac{1}{10}\end{aligned}$$

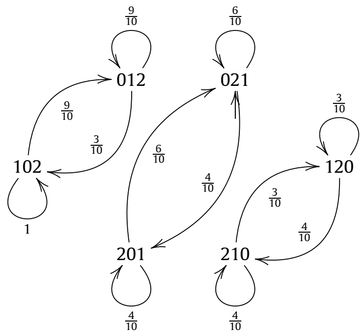

Fig. 4. Model $\mathcal{H}|_{(\mathbf{0_a} \vee \mathbf{2_a})}$: **c**lara's beliefs after the announcement $(\mathbf{0_a} \vee \mathbf{2_a})$.

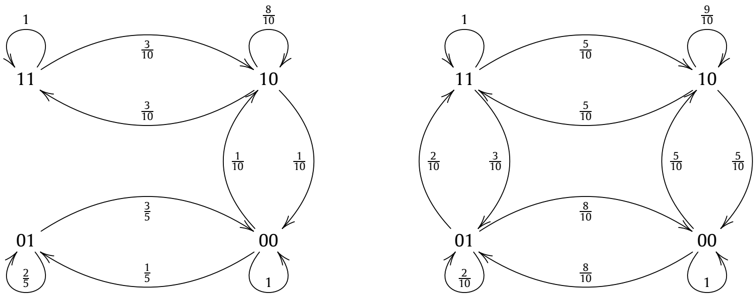

Fig. 5. Model $\mathcal{M}$: robots r_a and r_b beliefs.

Example 3 *(Controlling a Reservoir).* Let us consider a situation where two robots r_a and r_b are responsible for opening and closing two taps (t_a and t_b) in order to not allow the reservoir to overflow or to become empty. Tap t_a is controlled by r_a and tap t_b by r_b. The robots cannot communicate between them. The system can be in four states:

- **11** - both taps are closed;
- **10** - t_a is closed and t_b is open;
- **01** - t_a is open and t_b is closed;
- **00** - both taps are open.

Both robots have sensors for their perceptions of the state of the taps. However, the accuracy of their sensors is not the same: robot r_a has a more precise sensor and can percept the taps states with a finer gran set of confidencedegrees than r_b. We use the following fuzzy variables representing the value read by the robots:

- $\mathbf{1_a}$ - the reading of the sensor of robot r_a;
- $\mathbf{1_b}$ - the reading of the sensor of robot r_b.

The edges represent the degree of certainty that each robot has in each states. For instance, at state 10, robot r_a "thinks" it is indeed at state 10 with a degree of certainty of $\frac{8}{10}$ and it conceives that it could be at state 11 with degree $\frac{3}{10}$ and at state 00 with grade $\frac{1}{10}$.

For the sake of clarity, some edges are omitted in Fig. 5. All edges are presented by a matrix as follows:

R_{r_a}	11	10	00	01
11	1	$\frac{3}{10}$	0	0
10	$\frac{3}{10}$	$\frac{8}{10}$	$\frac{1}{10}$	0
00	0	$\frac{1}{10}$	1	$\frac{2}{10}$
01	0	0	$\frac{6}{10}$	$\frac{4}{10}$

R_{r_b}	11	10	00	01
11	1	$\frac{5}{10}$	0	$\frac{3}{10}$
10	$\frac{5}{10}$	$\frac{9}{10}$	$\frac{5}{10}$	0
00	0	$\frac{5}{10}$	1	$\frac{8}{10}$
01	$\frac{2}{10}$	0	$\frac{8}{10}$	$\frac{2}{10}$

The valuation function for each fuzzy variable at each state is

V	11	10	01	00
$\mathbf{1_a}$	$\frac{8}{10}$	$\frac{6}{10}$	$\frac{4}{10}$	$\frac{2}{10}$
$\mathbf{1_b}$	$\frac{6}{10}$	$\frac{6}{10}$	$\frac{3}{10}$	$\frac{1}{10}$

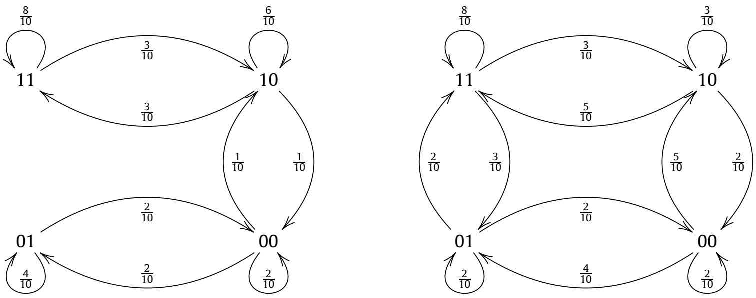

Fig. 6. Model $\mathcal{M}|_{\mathbf{1_a}}$: robots r_a and r_b beliefs after the announcement is performed.

As an illustration, we can evaluate the knowledge of the robots at state 10 about the situation of the taps.

$$\begin{aligned}
10 \models K_{r_a}(\mathbf{1_a} \wedge \mathbf{1_b}) = \;& \bigwedge \big\{ I(R_{r_a}(10,10), 10 \models (\mathbf{1_a} \wedge \mathbf{1_b})), I(R_{r_a}(10,00), 00 \models (\mathbf{1_a} \wedge \mathbf{1_b})), \\
& I(R_{r_a}(10,11), 11 \models (\mathbf{1_a} \wedge \mathbf{1_b})), I(R_{r_a}(10,01), 01 \models (\mathbf{1_a} \wedge \mathbf{1_b})) \big\} \\
= \;& \bigwedge \{ I(\tfrac{8}{10}, \tfrac{6}{10}), I(\tfrac{1}{10}, \tfrac{1}{10}), I(\tfrac{3}{10}, \tfrac{6}{10}), I(0, \tfrac{3}{10}) \} = \bigwedge \{ \tfrac{6}{10}, 1, 1, 1 \} = \tfrac{6}{10} \\
10 \models K_{r_b}(\mathbf{1_a} \wedge \mathbf{1_b}) = \;& \bigwedge \big\{ I(R_{r_b}(10,10), 10 \models (\mathbf{1_a} \wedge \mathbf{1_b})), I(R_{r_b}(10,00), 00 \models (\mathbf{1_a} \wedge \mathbf{1_b})), \\
& I(R_{r_b}(10,11), 11 \models (\mathbf{1_a} \wedge \mathbf{1_b})), I(R_{r_b}(10,01), 01 \models (\mathbf{1_a} \wedge \mathbf{1_b})) \big\} \\
= \;& \bigwedge \{ I(\tfrac{9}{10}, \tfrac{6}{10}), I(\tfrac{5}{10}\, \tfrac{1}{10}), I(\tfrac{5}{10}, \tfrac{6}{10}), I(0, \tfrac{3}{10}) \} = \bigwedge \{ \tfrac{6}{10}, \tfrac{1}{10}, 1, 1 \} = \tfrac{1}{10}
\end{aligned}$$

Now, it is announced the formula $\mathbf{1_a}$, at state 10, and the robots need to computes the formulas $K_{r_a}(\mathbf{1_a} \wedge \mathbf{1_b})$ and $K_{r_b}(\mathbf{1_a} \wedge \mathbf{1_b})$ after the announcement, i.e. they evaluate $10 \models [\mathbf{1_a}]K_{r_a}(\mathbf{1_a} \wedge \mathbf{1_b})$ and $10 \models [\mathbf{1_a}]K_{r_b}(\mathbf{1_a} \wedge \mathbf{1_b})$ (Fig. 6).

$$\mathcal{M}, 10 \models [\mathbf{1_a}]K_a(\mathbf{1_a} \wedge \mathbf{1_b}) = \quad I\big((\mathcal{M}, 10 \models \mathbf{1_a}), (\mathcal{M}|_{\mathbf{1_a}}, 10 \models K_a(\mathbf{1_a} \wedge \mathbf{1_b}))\big).$$

First, we have to compute the model $\mathcal{M}|_{\mathbf{1_a}}$ and then evaluate the formula $K_a(\mathbf{1_a} \wedge \mathbf{1_b})$ in this new model.

$$\begin{aligned}
\mathcal{M}|_{\mathbf{1_a}}, 10 \models K_{r_a}(\mathbf{1_a} \wedge \mathbf{1_b}) = \;& \bigwedge \big\{ I(R_{r_a}(10,10), \mathcal{M}|_{\mathbf{1_a}}, 10 \models (\mathbf{1_a} \wedge \mathbf{1_b})), I(R_{r_a}(10,00), \mathcal{M}|_{\mathbf{1_a}}, 00 \models (\mathbf{1_a} \wedge \mathbf{1_b})), \\
& I(R_{r_a}(10,11), \mathcal{M}|_{\mathbf{1_a}}, 11 \models (\mathbf{1_a} \wedge \mathbf{1_b})), I(R_{r_a}(10,01), \mathcal{M}|_{\mathbf{1_a}}, 01 \models (\mathbf{1_a} \wedge \mathbf{1_b})) \big\} \\
= \;& \bigwedge \{ I(\tfrac{6}{10}, \tfrac{6}{10}), I(\tfrac{1}{10}, \tfrac{1}{10}), I(\tfrac{3}{10}, \tfrac{6}{10}), I(0, \tfrac{3}{10}) \} = \bigwedge \{1, 1, 1, 1\} = 1 \\
\mathcal{M}|_{\mathbf{1_a}}, 10 \models K_{r_b}(\mathbf{1_a} \wedge \mathbf{1_b}) = \;& \bigwedge \big\{ I(R_{r_b}(10,10), \mathcal{M}|_{\mathbf{1_a}}, 10 \models (\mathbf{1_a} \wedge \mathbf{1_b})), I(R_{r_b}(10,00), \mathcal{M}|_{\mathbf{1_a}}, 00 \models (\mathbf{1_a} \wedge \mathbf{1_b})), \\
& I(R_{r_b}(10,11), \mathcal{M}|_{\mathbf{1_a}}, 11 \models (\mathbf{1_a} \wedge \mathbf{1_b})), I(R_{r_b}(10,01), \mathcal{M}|_{\mathbf{1_a}}, 01 \models (\mathbf{1_a} \wedge \mathbf{1_b})) \big\} \\
= \;& \bigwedge \{ I(\tfrac{3}{10}, \tfrac{6}{10}), I(\tfrac{2}{10}, \tfrac{1}{10}), I(\tfrac{5}{10}, \tfrac{6}{10}), I(0, \tfrac{3}{10}) \} = \bigwedge \{1, \tfrac{1}{10}, 1, 1\} = \tfrac{1}{10}
\end{aligned}$$

After the announcement we have

$$\begin{aligned}
\mathcal{M}, 10 \models [\mathbf{1_a}]K_{r_a}(\mathbf{1_a} \wedge \mathbf{1_b}) = \;& I\big((\mathcal{M}, 10 \models \mathbf{1_a}), (\mathcal{M}|_{\mathbf{1_a}}, 10 \models K_{r_a}(\mathbf{1_a} \wedge \mathbf{1_b}))\big), \\
= \;& I\big(\tfrac{6}{10}, 1\big) = 1 \\
\mathcal{M}, 10 \models [\mathbf{1_a}]K_{r_b}(\mathbf{1_a} \wedge \mathbf{1_b}) = \;& I\big((\mathcal{M}, 10 \models \mathbf{1_a}), (\mathcal{M}|_{\mathbf{1_a}}, 10 \models K_{r_b}(\mathbf{1_a} \wedge \mathbf{1_b}))\big), \\
= \;& I\big(\tfrac{6}{10}, \tfrac{1}{10}\big) = \tfrac{1}{10}
\end{aligned}$$

3.2. Properties

This section shows that fuzzy epistemic logic with public announcement respect the principles of the standard epistemic logic with public announcement. On this view, firstly we discuss that fuzzy epistemic logic generalises $\mathcal{ELP}$. Then, we show that all the axioms presented in section 2 are valid in all the fuzzy epistemic models.

Next result emphasizes that $\mathcal{FELP}$ generalises $\mathcal{ELP}$ in the sense that, whenever the *fuzzy epistemic models are crisp*, i.e., when they are of for $M = (W, R, V)$ such that

- for any $w, v \in W$, $R(w, w') \in \{0, 1\}$, and
- for any $w \in W$, $p \in \text{Prop}$, $V(w, p) \in \{0, 1\}$,

the graded satisfaction relation $\models$ coincides with the satisfaction $\models^{\mathcal{EPL}}$.

Theorem 1. *Let $M = (W, R, V)$ be a fuzzy crispy epistemic model. Then for any formula φ:*

1. $(M, w \models \varphi) \in \{0, 1\}$
2. $(M, w \models \varphi) = 1$ *iff* $M, w \models^{\mathcal{ELP}} \varphi$

Proof. The proof of *1.* is obvious.

In order to prove *2.*, let us consider formulas of the form $[\chi]\varphi$ (the proof for the other cases is straightforward).

First, note that $M|_\chi$ is always a fuzzy crispy epistemic model.

Let us suppose $(M, w \models [\chi]\varphi) = 1$, i.e. $I((M, w \models \chi), (M|_\chi, w \models \varphi)) = 1$. Hence,

- If $(M|_\chi, w \models \varphi) = 1$ then, by I.H., $M|_\chi, w \models^{\mathcal{ELP}} \varphi$. Therefore, $M, w \models^{\mathcal{ELP}} [\chi]\varphi$.
- If $(M|_\chi, w \models \varphi) = 0$, then $(M, w \models \chi) = 0$. Hence, by I.H. $M, w \not\models^{\mathcal{ELP}} \chi$. Therefore, $M, w \models^{\mathcal{ELP}} [\chi]\varphi$.

For the converse implication, let us suppose that $M \models^{\mathcal{ELP}} [\chi]\varphi$.

- If $M, w \not\models^{\mathcal{ELP}} \chi$ then, by I.H., $(M, w \models \chi) = 0$. Hence, $I\big((M, w \models \chi), (M|_\chi, w \models \varphi)\big) = 1$.
- If $M, w \models^{\mathcal{ELP}} \chi$ then, by I.H. $(M, w \models \chi) = 1$ and $(M|_\chi \models \varphi) = 1$. Hence, $I\big((M, w \models \chi), (M|_\chi, w \models \varphi)\big) = 1$. □

Theorem 2. *The properties*

A) *All instantiations of propositional tautologies,*
B) $K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$

are valid in any model.

Proof. The proof for A) and B) can be extracted from Lemma 9 of [16], since the connectives interpretation of our logic are that of a Gödel algebra, a reduct of an instantiation of an *action lattice*. □

It remains to study the axioms that distinguish epistemic logic from other modal logics. Different from previous theorem, the properties of the graded versions of reflexivity, symmetry and transitivity properties imposed in R, will be necessary:

Theorem 3. *The axioms*

C) $K_a\varphi \to \varphi$,
D) $K_a\varphi \to K_aK_a\varphi$,
E) $\varphi \to K_aB_a\varphi$

are valid in any fuzzy epistemic model.

Proof. In order to prove that for any fuzzy epistemic model $\mathcal{M}$ and for any $w \in W$, $(M, w \models K_a\varphi \to \varphi) = 1$, we have by (9) that it is enough to see that $(\mathcal{M}, w \models K_a\varphi) \leq (\mathcal{M}, w \models \varphi)$. For that, we observe that:

$$
\begin{aligned}
& (\mathcal{M}, w \models K_a\varphi) \\
= \quad & \{ \models \text{defn.} \} \\
& \bigwedge_{w' \in W} \{I(R_a(w, w'), (\mathcal{M}, w' \models \varphi))\} \\
\leq \quad & \{ \text{minimum properties} \} \\
& I(R_a(w, w), (\mathcal{M}, w \models \varphi)) \\
= \quad & \{ (4) \} \\
& I(1, (\mathcal{M}, w \models \varphi)) \\
= \quad & \{ \text{I defn.} \} \\
& (\mathcal{M}, w \models \varphi)
\end{aligned}
$$

To prove the validity of $K_a\varphi \to K_aK_a\varphi$, by (6) we have that

$$\text{for any } w' \in W, R_a(w, w'') \geq \min(R_a(w, w'), R_a(w', w''))$$

$\Leftrightarrow$ { min properties }

for any $w' \in W$, $R_a(w, w'') \geq \min(R_a(w', w''), R_a(w, w'))$

$\Rightarrow$ { (15) }

for any $w' \in W$, $I(R_a(w, w''), (\mathcal{M}, w'' \models \varphi)) \leq I\big((\min(R_a(w', w''), R_a(w, w'))), (\mathcal{M}, w'' \models \varphi)\big)$

$\Leftrightarrow$ { infimum properties }

$$I\big(R_a(w, w''), (\mathcal{M}, w'' \models \varphi)\big) \leq \bigwedge_{w' \in W} \{I\big(\min(R_a(w', w''), R_a(w, w')), (\mathcal{M}, w'' \models \varphi)\big)\}$$

$\Leftrightarrow$ { (10) }

$$I\big(R_a(w, w''), (\mathcal{M}, w'' \models \varphi)\big) \leq \bigwedge_{w' \in W} \{I(R_a(w, w'), I((R_a(w', w''), (\mathcal{M}, w'' \models \varphi))))\}$$

$\Rightarrow$ { inf. monotocity }

$$\bigwedge_{w'' \in W} \{I\big(R_a(w, w''), (\mathcal{M}, w'' \models \varphi)\big)\} \leq \bigwedge_{w', w'' \in W} \{I\big(R_a(w, w'), I(R_a(w', w''), (\mathcal{M}, w'' \models \varphi))\}$$

$\Leftrightarrow$ { (12) }

$$\bigwedge_{w'' \in W} \{I\big(R_a(w, w''), (\mathcal{M}, w'' \models \varphi)\big)\} \leq \bigwedge_{w' \in W} \{I\big(R_a(w, w'), \bigwedge_{w'' \in W} \{I(R_a(w', w''), (\mathcal{M}, w'' \models \varphi))\}\big)\}$$

$\Leftrightarrow$ { $\models$ defn. twice }

$(\mathcal{M}, w \models K_a\varphi) \leq (\mathcal{M}, w \models K_a K_a \varphi)$

Finally, we show that for any model $\mathcal{M}$ and for any state w, $(M, w \models K_a B_a \varphi) \geq (M, w \models \varphi)$, as follows:

$(\mathcal{M}, w \models K_a B_a \varphi)$

$=$ { $\models$ defn. }

$$\bigwedge_{w' \in W} \{I(R(w, w'), (\mathcal{M}, w' \models B_a\varphi)\}$$

$=$ { $\models$ defn. }

$$\bigwedge_{w' \in W} \{I(R(w, w'), \bigvee_{w'' \in W} \{\min\{(R(w', w''), (\mathcal{M}, w'' \models \varphi)\}\}\}$$

$\geq$ { (14) + $\bigvee_{w'' \in W} \{\min\{(R(w', w''), (\mathcal{M}, w'' \models \varphi)\} \geq \min\{(R(w', w), (\mathcal{M}, w \models \varphi)\}\}$

$$\bigwedge_{w' \in W} \{I(R(w, w'), \min\{(R(w', w), (\mathcal{M}, w \models \varphi)\}\}$$

$=$ { (5) }

$$\bigwedge_{w' \in W} \{I(R(w, w'), \min\{(R(w, w'), (\mathcal{M}, w \models \varphi)\}\}$$

$=$ { (12) }

$$\bigwedge_{w' \in W} \{I(R(w, w'), (\mathcal{M}, w \models \varphi))\}$$

$=$ { (13) }

$$I\Big(\bigvee_{w' \in W} \{R(w, w')\}, (\mathcal{M}, w \models \varphi)\Big)$$

$\geq$ { (15) and (4) }

$I\big(1, (\mathcal{M}, w \models \varphi)\big)$

$=$ { defn. I}

$(\mathcal{M}, w \models \varphi)$ □

Theorem 4. *The axioms*

F) $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$

G) $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
H) $[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$
I) $[\varphi][\psi]\chi \leftrightarrow \big[\varphi \wedge [\varphi]\psi\big]\chi$
J) $[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$

are valid in any fuzzy epistemic model.

Proof. Firstly, we observe that it is true for any φ, ψ that

$$(\mathcal{M}, w \models \varphi \rightarrow \psi) = 1 \text{ iff } (\mathcal{M}, w \models \varphi) = (\mathcal{M}, w \models \psi) \tag{17}$$

Since

$$\begin{aligned}
&(\mathcal{M}, w \models \varphi \leftrightarrow \psi) = 1 \\
\Leftrightarrow \quad & \{ \leftrightarrow \text{defn.} \} \\
&(\mathcal{M}, w \models (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)) = 1 \\
\Leftrightarrow \quad & \{ \models \text{defn.} \} \\
&\min\{(\mathcal{M}, w \models (\varphi \rightarrow \psi)), (\mathcal{M}, w \models \psi \rightarrow \varphi)\} = 1 \\
\Leftrightarrow \quad & \{ 1 \text{ is the largest element } \} \\
&(\mathcal{M}, w \models (\varphi \rightarrow \psi)) = 1 \text{ and } (\mathcal{M}, w \models \psi \rightarrow \varphi) = 1 \\
\Leftrightarrow \quad & \{ \models \text{defn + (9) twice } \} \\
&(\mathcal{M}, w \models \varphi) \leq (\mathcal{M}, w \models \psi)) \text{ and } (\mathcal{M}, w \models \psi) \leq (\mathcal{M}, w \models \varphi) \\
\Leftrightarrow \quad & \{ \text{linearity of } \leq \} \\
&(\mathcal{M}, w \models \varphi) = (\mathcal{M}, w \models \psi)
\end{aligned}$$

Hence, we observe that

$$\begin{aligned}
&(\mathcal{M}, w \models [\varphi]p) \\
= \quad & \{ \models \text{defn.} \} \\
&I\big((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models p)\big) \\
= \quad & \{ \mathcal{M}|_\varphi \text{ defn.} \} \\
&I\big((\mathcal{M}, w \models \varphi), (M, w \models p)\big) \\
= \quad & \{ \models \text{defn.} \} \\
&(\mathcal{M}, w \models \varphi \rightarrow p)
\end{aligned}$$

Hence, by (17) we have that $(\mathcal{M}, w \models [\varphi]p \leftrightarrow (\varphi \rightarrow p)) = 1$. Analogously, we observe that

$$\begin{aligned}
&(\mathcal{M}, w \models [\varphi](\psi \wedge \chi)) \\
= \quad & \{ \models \text{defn.} \} \\
&I\big((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi \wedge \chi)\big) \\
= \quad & \{ \models \text{defn.} \} \\
&I\big((\mathcal{M}, w \models \varphi), \min\{(\mathcal{M}|_\varphi, w \models \psi), (\mathcal{M}|_\varphi, w \models \chi)\}\big) \\
= \quad & \{ (12) \} \\
&\min\big\{I((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi)), I((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \chi))\big\} \\
= \quad & \{ \models \text{defn.} \} \\
&\min\big\{(\mathcal{M}, w \models [\varphi]\psi), (\mathcal{M}, w \models [\varphi]\chi)\big\} \\
= \quad & \{ \models \text{defn.} \} \\
&(\mathcal{M}, w \models [\varphi]\psi \wedge [\varphi]\chi)
\end{aligned}$$

to prove by (17) that $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$. Following the same strategy, to prove the validity $[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$ as follows:

$$
\begin{aligned}
& (\mathcal{M}, w \models [\varphi]K_a\psi) \\
= & \quad \{ \models \text{defn.} \} \\
& I\big((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models K_a\psi)\big) \\
= & \quad \{ \models \text{defn.} \} \\
& I\big((\mathcal{M}, w \models \varphi), \bigwedge_{w' \in W} \big\{I\big((R|_\varphi)_a(w, w'), (\mathcal{M}|_\varphi, w' \models \psi)\big)\big\}\big) \\
= & \quad \{ R|_\varphi \text{ defn.} \} \\
& I\big((\mathcal{M}, w \models \varphi), \bigwedge_{w' \in W} \big\{I\big(\min\{R_a(w, w'), (M, w' \models \varphi)\}, (\mathcal{M}|_\varphi, w' \models \psi)\big)\big\}\big) \\
= & \quad \{ (10) \} \\
& I\big((M, w \models \varphi), \bigwedge_{w' \in W} \big\{I\big(R_a(w, w'), I((\mathcal{M}, w' \models \varphi), (\mathcal{M}|_\varphi, w' \models \psi))\big)\big\}\big) \\
= & \quad \{ \models \text{defn.} \} \\
& I\big((\mathcal{M}, w \models \varphi), \bigwedge_{w' \in W} \big\{I\big(R_a(w, w'), (\mathcal{M}, w' \models [\varphi]\psi)\big)\big\}\big) \\
= & \quad \{ \models \text{defn.} \} \\
& (\mathcal{M}, w \models \varphi \rightarrow K_a[\varphi]\psi)
\end{aligned}
$$

Finally, we prove the validity of $[\varphi][\psi]\chi \leftrightarrow \big[\varphi \wedge [\varphi]\psi\big]\chi$ by observing, on the one hand that:

$$
\begin{aligned}
& (\mathcal{M}, w \models [\varphi][\psi]\chi) \\
= & \quad \{ \models \text{defn.} \} \\
& I\big((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models [\psi]\chi)\big) \\
= & \quad \{ \models \text{defn.} \} \\
& I\big((\mathcal{M}, w \models \varphi), I\big((\mathcal{M}|_\varphi, w \models \psi), ((\mathcal{M}|_\varphi)|_\psi, w \models \chi)\big)\big)
\end{aligned}
$$

and, on the other hand that

$$
\begin{aligned}
& (\mathcal{M}, w \models \big[\varphi \wedge [\varphi]\psi\big]\chi) \\
= & \quad \{ \models \text{defn.} \} \\
& I\big((\mathcal{M}, w \models \varphi \wedge [\varphi]\psi), (\mathcal{M}|_{\varphi \wedge [\varphi]\psi}, w \models \chi)\big) \\
= & \quad \{ \models \text{defn.} \} \\
& I\big(\min\{(\mathcal{M}, w \models \varphi), (\mathcal{M}, w \models [\varphi]\psi)\}, (\mathcal{M}|_{\varphi \wedge [\varphi]\psi}, w \models \chi)\big) \\
= & \quad \{ \models \text{defn.} \} \\
& I\big(\min\{(\mathcal{M}, w \models \varphi), I\big((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi)\big)\}, (\mathcal{M}|_{\varphi \wedge [\varphi]\psi}, w \models \chi)\big) \\
= & \quad \{ (11) \} \\
& I\big(\min\{(\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi)\}, (\mathcal{M}|_{\varphi \wedge [\varphi]\psi}, w \models \chi)\big) \\
= & \quad \{ (10) \} \\
& I\Big((\mathcal{M}, w \models \varphi), I\big((\mathcal{M}|_\varphi, w \models \psi), (\mathcal{M}|_{\varphi \wedge [\varphi]\psi}, w \models \chi)\big)\Big)
\end{aligned}
$$

Hence, in order to see $(\mathcal{M}, w \models [\varphi][\psi]\chi) = (\mathcal{M}, w \models \big[\varphi \wedge [\varphi]\psi\big]\chi)$ it is enough to prove

$$\mathcal{M}|_{\varphi \wedge [\varphi]\psi} = (\mathcal{M}|_\varphi)|_\psi \tag{18}$$

i.e. that for any $a \in \text{Ag}$, $(R|_{\varphi \wedge [\varphi]\psi})_a = ((R|_\varphi)|_\psi)_a$. To prove such an equality, let us observe that for any $w, w' \in W$,

$$
\begin{aligned}
& (R|_{\varphi\wedge[\varphi]\psi})_a(w,w') \\
= \quad & \{ \text{ models update defn.} \} \\
& \min\{R|_a(w,w'), (\mathcal{M}, w \models \varphi \wedge [\varphi]\psi)\} \\
= \quad & \{ \models \text{ defn. } \} \\
& \min\left\{R|_a(w,w'), \min\{(\mathcal{M}, w \models \varphi), (\mathcal{M}, w \models [\varphi]\psi)\}\right\} \\
= \quad & \{ \models \text{ defn. } \} \\
& \min\left\{R_a(w,w'), \min\{(\mathcal{M}, w \models \varphi), I\big((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi)\big)\}\right\} \\
= \quad & \{ \text{ (11) } \} \\
& \min\left\{R_a(w,w'), \min\{(\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi)\}\right\} \\
= \quad & \{ \text{ min assoc. } \} \\
& \min\left\{\min\{R_a(w,w'), (\mathcal{M}, w \models \varphi)\}, (\mathcal{M}|_\varphi, w \models \psi)\right\} \\
= \quad & \{ \text{ models update defn. } \} \\
& \min\left\{(R|_\varphi)_a(w,w'), (\mathcal{M}|_\varphi, w \models \psi)\right\} \\
= \quad & \{ \text{ models update defn. } \} \\
& ((R|_\varphi)|_\psi)_a(w,w')
\end{aligned}
$$

Finally, let us prove the validity of axiom $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$. Let us consider the values

- $\alpha = (\mathcal{M}, w \models [\varphi]\neg\psi) = I\big((\mathcal{M}, w \models \varphi), N(\mathcal{M}|_\varphi, w \models \psi)\big)$ and
- $\beta = (\mathcal{M}, w \models \varphi \to \neg[\varphi]\psi) = I\big((\mathcal{M}, w \models \varphi), N\big(I\big((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi)\big)\big)\big)$.

Now, we prove $\alpha = \beta$.
Case $(\mathcal{M}, w \models \varphi) = 0$, by (7) we have $\alpha = 1 = \beta$.
Case $(\mathcal{M}, w \models \varphi) \neq 0$, by (8) we have that $N(\mathcal{M}|_\varphi, w \models \psi) = 0$ and, by (7), $\alpha = 0$. Thus,

- if $(\mathcal{M}, w \models \varphi) \leq (\mathcal{M}|_\varphi, w \models \psi)$, by (8) and (7), we have $N\big(I((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi))\big) = 0$. Since $(\mathcal{M}, w \models \varphi) \neq 0$, we have, by (7), that $\beta = 0$.
- if $(\mathcal{M}, w \models \varphi) > (\mathcal{M}|_\varphi, w \models \psi)$, by (7), $I((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi)) = (\mathcal{M}|_\varphi, w \models \psi)$. Since $(\mathcal{M}, w \models \varphi) \neq 0$, $N(\mathcal{M}, w \models \varphi) = 0$. Hence, $\beta = 0$. □

4. Bisimulation and modal invariance

It is well known that suitable notions of bisimulation, i.e. behaviour preserving maps between models, assume a crucial role in any modal logic. On the logic view, they provide a map that preserves the satisfaction of formulas - the so-called modal invariance property. For practical purposes, it provides the way of identify models that are indistinguishable by behaviours and by logical properties, and paves the way for the implementation of reduction techniques based on quotients. This section introduces a bisimulation notion for our logic and proves that it enjoys of the modal invariance property.

Definition 5 *(Bisimulation).* Let $\mathcal{M} = (W, R, V)$, $\mathcal{M}' = (W', R', V')$ be two (Prop, A)-models and $B \subseteq W \times W'$ a binary relation. We say that B is a *bisimulation from* $\mathcal{M}$ *to* $\mathcal{M}'$ if, for every $w \in W$ and $w' \in W'$ such that $(w, w') \in B$, we have

(Atoms) for any $p \in \text{Prop}$, $V(w, p) = V'(w', p)$
(Fzig) for any $v \in W$, $R(w, v) \leq \bigvee_{v' \in B[v]} R'(w', v')$, where $B[v] = \{v' \in W' |\, (v, v') \in B\}$.
(Fzag) for any $v' \in V'$, $R'(w', v') \leq \bigvee_{v \in B^{-1}[u']} R(w, v)$, where $B^{-1}[v'] = \{v \in W |\, (v, v') \in B\}$

As mentioned, we establish the modal invariance property for this logic:

Theorem 5. *Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be two* (Prop, A)*-models and $B \subseteq W \times W'$ a bisimulation from $\mathcal{M}$ to $\mathcal{M}'$. Then, for any $w \in W$ and $w' \in W'$ such that $(w, w') \in B$, we have for any formula $\rho \in \text{Fm}^{\mathcal{FELP}}(\text{Prop}, A)$, that*

$$(\mathcal{M}, w \models \rho) = (\mathcal{M}', w' \models \rho)$$

Proof. The proof is given by induction on the structure of the formulas. Most of the cases can be straightforward obtained from the invariance of Fuzzy Modal Logics provided in [11]. Now we will prove the invariance for ρ of form $[\varphi]\psi$. First, observe that

$$\mathcal{M} \sim \mathcal{M}' \text{ implies that } \mathcal{M}|_\varphi \sim \mathcal{M}'|_\varphi \tag{19}$$

In order to prove **(Fzig)** of B w.r.t. $\mathcal{M}|_\varphi$ and $\mathcal{M}'|_\varphi$, we observe that:

$$R|_\varphi(w,v) \leq \bigvee_{v' \in B[v]} R'|_\varphi(w',v') \tag{20}$$

$$\begin{aligned}
& R|_\varphi(w,v) \\
= \quad & \{ \text{ models update defn. } \} \\
& \min(R(w,v), (\mathcal{M}, v \models \varphi)) \\
\leq \quad & \{ \text{ (zig) of } B\text{, monotonicity of min } \} \\
& \min\Big(\bigvee_{v' \in B[v]} \{R'(w',v')\}, (\mathcal{M}, v \models \varphi)\Big) \\
\leq \quad & \{ \; W' \text{ finite + (16) } \} \\
& \bigvee_{u' \in B[v]} \big\{ \min(R'(w',v'), (\mathcal{M}, v \models \varphi)) \big\} \\
= \quad & \{ \text{ I.H. } \} \\
& \bigvee_{v' \in B[v]} \big\{ \min(R'(w',v'), (\mathcal{M}', v' \models \varphi)) \big\} \\
= \quad & \{ \text{ models update defn. } \} \\
& \bigvee_{v' \in B[v]} \{R'|_\varphi(w',v')\}
\end{aligned}$$

The **(Fzag)** can be obtained similarly and **(Atom)** trivially holds. Therefore, for public announcements $[\varphi]\psi$:

$$\begin{aligned}
& (\mathcal{M}, w \models [\varphi]\psi) \\
= \quad & \{ \models \text{ defn. } \} \\
& I\big((\mathcal{M}, w \models \varphi), (\mathcal{M}|_\varphi, w \models \psi))\big) \\
= \quad & \{ \text{ (19) + I.H. twice } \} \\
& I\big((\mathcal{M}', w' \models \varphi), (\mathcal{M}'|_\varphi, w' \models \psi))\big) \\
= \quad & \{ \models \text{ defn. } \} \\
& (\mathcal{M}', w' \models [\varphi]\psi) \quad \square
\end{aligned}$$

5. Conclusions and related work

This work introduces a new fuzzy epistemic logic with public announcement endowed with a bisimulation notion that is invariant for the introduced $\mathcal{FELP}$. Moreover, we studied the soundness of the axioms of the standard bivalent version, and we illustrate it with some demonstrative examples. There is a myriad of research lines worth to pursue.

As we did in a more generic context in [2], the common knowledge modalities can be naturally adjusted to such a setting. In this view, we belief that this work can be extended with an adapted notion of common knowledge modalities. Moreover, the accommodation of other more complex epistemic interactions, including private and suspicious announcements, as in the action models [1], is also in our agenda.

Proof support for this logic is naturally a main concern. As showed above, the axioms of the standard epistemic logic with public announcement are sound for the presented fuzzy semantics. This axiomatization is a starting point candidate for a proof calculi for the logic. Its completeness, however, is still an open question. Once the public announcement axioms are all sound in our semantics, we could use them as reduction axioms in order to prove that every formula in our logic with public announcement is equivalent to a formula in Multi-agent Epistemic Logics (see [4,5]). The completeness of our logic would then follow naturally from the completeness of the latter.

Related work. Reference [7] studied Multi-Agent Epistemic Logics to represent and reason about knowledge and beliefs of agents or groups of agents. A number of other proposals also extend these logics with support to uncertainty, as the are cases of [9,18,21,14]. Some extensions aimed to reasoning about knowledge and probabilities [6] by extending the language with weighted formulas and adding probabilities to the semantics.

In [2], the authors propose a parametric method to construct Multi-valued Epistemic Logics (public announcements were not considered). Such a method follows the lines of the systematic generation of Multi-valued Dynamic Logics introduced in [16,17]. These methods are parametric to Action Lattices [13], to support both a generic space of agent knowledge operators, as choice, composition and closure (as a Kleene algebra), and a proper truth space for possible non bivalent interpretation of the assertions (as a residuated lattice).

Other versions of public announcements in multi-valued setting were explored in the literature. In [21], it is proposed a four-value PAL where formulas are evaluated in four-valued Kripke models which assign to propositions and relations one of the four values. Another related paper is [3], in which is developed a public announcement logic based on finite-valued Lukasiewiscz modal logic. The semantics is based on many-valued Kripke models, where relations are crisp and propositions and formulas are fuzzy.

Finally, some notes about other Gödel algebra based Modal Logics (our semantics is based on $[0,1]$-valued Kripke models, where relation and formulas are $[0,1]$-valued). In [4], the authors prove strong completeness of the $\Box$-version ($G_\Box$) and the $\Diamond$-version ($G_\Diamond$) of a Gödel modal logic based on Kripke models where propositions at each world and the accessibility relation are both infinitely valued in the standard Gödel algebra $[0,1]$. In [5], it is shown that the full bi-modal logic based in Gödel-Kripke models is axiomatized by the system $G_{\Box\Diamond}$, which results from adding the union of $G_\Box$ and $G_\Diamond$ Fischer-Servi's connecting axioms. To extend the completeness theorem to the [0,1]-valued analogues of the classical bi-modal systems T, $S4$, $S5$ the authors introduced a particular kind of models, called optimal models. In [22], it is presented an axiomatization of crisp Gödel modal logic whose Kripke models have crisp accessibility and whose propositions are valued over the standard Gödel algebra. Finally, in [19] is proposed a Gödel modal logic with a $\Box_a$-modality for each value in a lattice $a \in \mathcal{A}$, meaning: $\Box_a \phi$ - the plausibility measure of ϕ is equal to a.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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